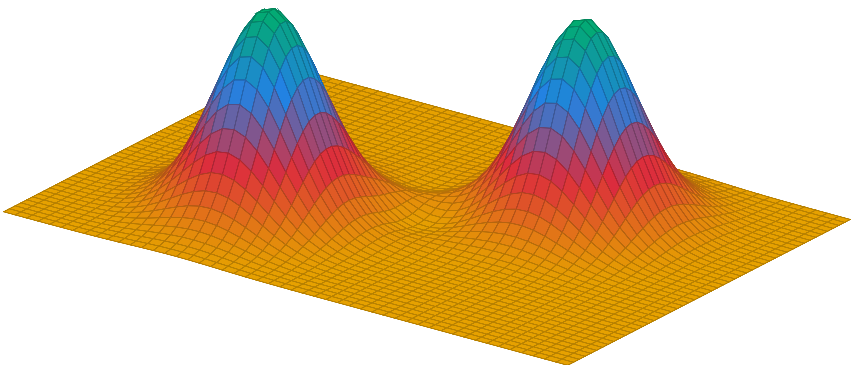
Master Thesis MSc Physics ETH

Transformations Between Imperfect Quantum Reference Frames



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Abstract

Starting from the point of view of an observer, we provide a new construction for unitary quantum reference frame transformations between observer perspectives, and under physical assumptions derive the existence of an observer-independent, *external view.* The non-trivial problem of reversibly transforming between physically relevant imperfect reference frames is solved by embedding such frames in perfect ones. Thanks to this embedding, our approach allows transforming into the perspective of an imperfect quantum reference frame, in a way which is consistent with the rich information theory of such frames. We explore the consequences of the embedding and explain the point of view of an observer whose frame is imperfect. The findings are applied to imperfect reference frames for one-dimensional Galilei transformations, in light of potential future applications in quantum gravity.

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Contents

1. Introduction

Any description of physics will inevitably require the notion of an *observer*, since we ourselves are observers. Closely linked to observers is the notion of a *reference frame*, intuitively the *point of view* an observer has on nature. Physical quantities are then understood *relative* to this reference frame. In classical physics, reference frames are often imagined as in principle realizable using physical objects [1, 2]; for instance, one may imagine how a Cartesian coordinate system, a reference frame for motion in space, could be constructed from a rigid body. It thus seems inevitable to also consider reference frames realizable from *quantum objects*, if one is studying quantum phenomena [3–5]. This idea is the main motivation behind the field of *quantum reference frames* and *quantum reference frame transformations*, which has recently received a lot of attention [6–15, and sources therein]. Compared to reference frames in classical physics, quantum reference frames can be in superpositions of orientations, and exhibit other quantum phenomena such as entanglement.

Quantum reference frames can be further divided into *perfect* and *imperfect* frames, based on how well they *break the symmetry* of the symmetry group relevant in the studied scenario. Intuitively, a reference frame breaks the symmetry perfectly, if it can perfectly keep track of every symmetry transformation which acts on the studied quantum system. For instance, the position of a classical particle is a perfect reference frame for translations, if position can be measured to arbitrary precision. Due to uncertainty relations, quantum objects occurring in nature *typically make imperfect reference frames.* For instance, one cannot perfectly localize a quantum particle without letting its momentum uncertainty and thus also energy uncertainty become unbounded; thus it makes an imperfect reference frame for translations. The difference between perfect and imperfect quantum reference frames is illustrated in figure 1.1.

Without the notion of transforming between frames, quantum frames are still immensely useful as references, relative to which quantum information, such as the orientation of a spin, can be understood; this idea has lead to the development of a rich information theory of imperfect reference frames [6, 16–24, and sources therein].

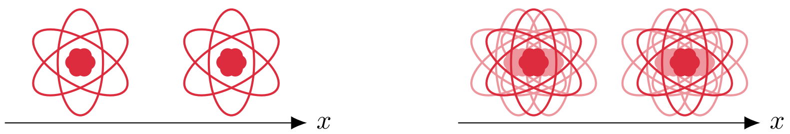


Figure 1.1: Left: An unrealistic, perfectly localizable quantum particle as a perfect reference frame for translations in one dimension. It may be in superposition, as indicated, or exhibit other quantum phenomena. Right: A more realistic quantum particle, which cannot be perfectly localized, as a reference frame for translations in one dimension. This particle too may be in superposition.

Importantly, quantum reference frames are also expected to play a role in the development of quantum gravity [10, 13–15, 25–27]. For instance, one may hope to better understand the gravitational field of a gravitating mass in superposition by transforming into the frame of the mass, where the gravitational field could be argued to become classical. And since quantum masses cannot be perfectly localized, we have reason to believe that especially imperfect reference frames will be relevant.

We are thus interested in transformations between imperfect quantum reference frames. Such transformations have however been studied less extensively than transformations between perfect frames (see *e.g.* [8, 15] for the latter). This stems in part from the difficulty if not impossibility of reconstructing the view of one observer from the view of another: an imperfect observer, say Alice, might not have access to all the information necessary to reconstruct the view of another observer, say Bob, leading to irreversible transformations (see *e.g.* [6]). We here take the standpoint that the views of both observers exist, and thus there must be a way of reversibly, *i.e.* unitarily transforming between them, even if this requires keeping track of information which might not be accessible to certain observers. We do not ask: "what does Alice think that Bob sees?" But rather: "what does Alice see and what does Bob see?" There already exists a framework capable of unitary transformations between imperfect reference frames, the so-called *perspective-neutral approach* [9–14]. Unfortunately, it relies on coherent methods (inspired by the *Page-Wootters formalism* for time evolution [28, 29]) and because of this is incompatible with the rich information theory of imperfect reference frames mentioned above. In short, the perspective-neutral approach uses a coherent group average (so-called *coherent G-twirl*) to handle the lack of knowledge of certain degrees of freedom in absence of a reference, while the mentioned information theory relies on an information-theoretically more natural incoherent average (the *G-twirl*) for the same purpose. The framework [15] uses the incoherent approach, and is thus compatible with the information theory of imperfect frames, but it cannot handle imperfect reference frames.

The goal of this thesis is to extend the formalism [15] to allow for transformations between imperfect reference frames, in light of potential applications in quantum gravity. The result is to our knowledge the first framework of quantum reference frame transformations which (1) produces unitary transformations between reference frames, (2) can deal with imperfect frames, and (3) is compatible with the information theory of imperfect frames. A detailed outline of how this is done as well as a list of results and contributions of the thesis is provided in the next section.

1.1 Outline

We begin by introducing *perfect* and *imperfect reference frames* in chapter 2, with emphasis on *quantum reference frames*. These notions have been firmly established in the literature; most of the chapter will thus be an introduction to already existing concepts. We will see that the defining feature of a reference frame is its ability to *break the symmetry of a group G*; whether this works perfectly distinguishes perfect from imperfect reference frames.

In chapter 3 we introduce the formalism [15] of reversible transformations between perfect reference frames. We do this in a novel way, by beginning with the point of view of an observer, and constructing the transformations from there. Among other things, we will derive the existence of an observer-independent *external view*, which (in a sense explained later) was the starting point for the original derivation of the formalism. Also, we will not consider the algebras of observables accessible by different observers studied in [15], but rather consider states which transform from one observer to another.

In chapter 4 we then extend the formalism described in chapter 3 to allow for reversible transformations between imperfect quantum reference frames. This is done through introducing *embeddings* of imperfect quantum reference frames into perfect ones. We observe that embedding of imperfect reference frame states into perfect ones is necessary to yield reversible transformations, but that this does not conflict with observers only having access to the imperfect states. We also describe the physical implications of such embeddings and explain what an observer in an imperfect quantum reference frame sees; notably, such an observer tends to have a "fuzzy view" of physics due to the imperfection of their frame.

Chapter 5 introduces the *Galilei transformations* in one dimension, those being among the symmetry groups which are interesting for quantum gravitational applications. Both the Galilei group and its *central extension*, as well as their mass-m representations, describing the action on quantum particles, are discussed. The central extension is relevant because it makes the mass-m representations non-projective and thus suitable for our formalism, and we will see that it is the natural group to consider when describing particles of potentially variable mass. We also discuss the representation theory of the centrally extended Galilei group.

Finally, we apply our new formalism to imperfect Galilei reference frames of quantum particles in chapter 6. We find that a single quantum particle always yields an imperfect quantum reference frame and argue that quantum particles in *squeezed coherent* states are an ideal choice for such reference frames. We close by describing the view of an observer in such a frame and illustrate the "fuzzy view" found earlier.

There are two main novel contributions in this thesis: Firstly, we provide a new, "observer-first" approach to the formalism of reversible transformations between perfect quantum reference frames of [15]. This new approach is less abstract than the original (the abstract approach has its own benefits however), and since it begins with the familiar view of an observer, it can serve as a pedagogical introduction to quantum reference frame transformations. Secondly, and perhaps more importantly, we provide a new way of handling reversible transformations between imperfect frames, which is compatible with the rich information theory of imperfect frames, since it does not rely on coherent techniques. This approach is enabled through the embedding of imperfect frames into perfect ones.

1.2 Notations and Conventions

We work in *natural units* where $\hbar = 1$.

A dot " $\cdot$ " instead of a function argument, *i.e.* "$(\,\cdot\,)$", "$[\,\cdot\,]$", "$(\,\cdot\,,\,\cdot\,)$", *etc.* stands for an unspecified function argument.

Hilbert spaces [30] are denoted by calligraphic letters, typically by $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_A$, *etc.* The inner product is $(\,\cdot\,,\,\cdot\,)$.

States $\psi \in \mathcal{H}$ in a Hilbert space are usually written in the *bra-ket notation* [30]: $\psi = |\psi\rangle$, $\langle\psi|\phi\rangle = (\psi, \phi)$, and $\langle\psi| = (\psi,\,\cdot\,)$.

Given a Hilbert space $\mathcal{H}$, $\overline{\mathcal{H}}$ denotes the larger vector space obtained in the context of *rigged Hilbert spaces* [30, 31]. $\overline{\mathcal{H}}$ is not a Hilbert space, but certain *improper* states in $\overline{\mathcal{H}}\backslash\mathcal{H}$ are compatible with the scalar product of $\mathcal{H}$ (generally having infinite normalizations), and we use braket-notation for those states as well.

Operators on Hilbert spaces, as well as their extensions to larger vector spaces in the context of rigged Hilbert spaces, are denoted with a caret: $\hat{A}$, $\hat{\rho}$, $\hat{U}$, *etc.*

G denotes a connected, non-trivial *Lie group* [32, 33]. Our Lie groups are finite-dimensional. Except for section 2.3, G will always be *unimodular* [34] (the left and right Haar measures coincide). We will always work with a fixed yet arbitrary (left and right) Haar measure. $|G|$ is the *total volume of the group* and can be infinite (if and only if G is non-compact, see theorem B.4). $|G|$ should not be confused with the *number of elements in G*, which we denote by $\#G$, and which is always uncountably infinite.

Except for section 2.2, all representations of G will be *unitary* and *non-projective* [32]. The mass-m representations $\hat{U}_m(a,v)$ of the Galilei group Gal in chapter 5 are projective. We will usually not specify that representations are non-projective, but always if they are.

When talking of *squeezed coherent* states of a quantum particle, we also mean the special case of *coherent* states (*i.e.* with no squeezing).

2. Perfect and Imperfect Reference Frames

Here we introduce the notion of a *reference frame for a group G*. Section 2.1 begins by describing so-called *perfect reference frames*, which intuitively can be used to *perfectly break the symmetry* of G by perfectly keeping track of G-transformations. Section 2.2 then specializes to perfect *quantum* reference frames, and section 2.3 provides an explicit construction of such quantum frames. Besides perfect reference frames, we will also be strongly interested in *imperfect reference frames*, which are introduced in section 2.4. Finally, section 2.5 discusses ways of comparing different imperfect frames with each other thanks to so-called *badness measures*.

2.1 Perfect Reference Frames

Intuitively, a *reference frame* is a physcial system used to keep track of *transformations* acting on a larger physical system containing the frame. The notion of transformations is most readily provided by a *group* (see *e.g.* [32] for the definition of a group). We will be interested in continuous transformations and thus consider any connected *Lie group G*. A Lie group is a group which is also a *smooth manifold* such that the group operation as well as the group inverse are smooth (see *e.g.* [32, 33] for detailed definitions). In the chapters 5 and 6 we will specialize to Galilei transformations in one dimension and thus consider variants of the one-dimensional Galilei group. For now, we will however keep our discussion general.

A reference frame should keep track of transformations in the sense that from two given reference frame states it should be possible to experimentally determine the unique transformation $g \in G$ which maps the first into the second. Or rather, that should at least be possible for a subset of all reference frame states. We also say, that a reference frame for G is a physical system which *perfectly breaks the symmetry of G*. This symmetry-breaking requirement is often taken, either explicitly or implicitly, as a basis for defining (quantum-) reference frames [10, 13–15, 19, 20].

Besides being a tool to break the symmetry of a group, reference frames in physics importantly also serve as a *point of view* which can be taken: it should be possible to link reference frames to *observers*. We will discuss this aspect of reference frames later, in chapter 3, and only for the case of *quantum reference frames* relevant for us.

Basic Notions. To make the idea of a reference frame precise, we begin by considering a *representation* of G acting on the space Σ of some physical system. Recall [32][1]:

Definition 2.1: Representation

Consider a set X on which G acts through a continuous *group action*

$$U : G \to \{X \to X\}, \tag{2.1}$$

i.e., U maps group elements to transformations on X. The action U is called a *representation* of the Lie group G, if it preserves the group multiplication:

$$U(g') \circ U(g) = U(g'g) \quad \forall g, g' \in G. \tag{2.2}$$

We now assume the existence of special states in Σ, the *classical reference frame states*, such that they implement the symmetry-breaking requirement. More precisely:

Definition 2.2: Perfect Reference Frame

Consider the state space Σ of a physical system, acted upon by a representation U of G. If there exists a subset of states $C \subset \Sigma$ with the properties

(a) the states C are *physically distinguishable* (see clarifications below),

(b) C is an *orbit* of G under U (*i.e.* G acts *transitively* on C),

(c) for any $\sigma, \sigma' \in C$ there exists precisely one $g \in G$ such that $\sigma' = U(g)(\sigma)$ (*i.e.* G acts *freely* on C),

then we say that the physical system is a *perfect reference frame for G* with the *classical reference frame states C*.

With "physically distinguishable" we mean that one can perform an experiment to determine which of the states in C the system is in, provided we know that it is in one of those states. The exact meaning depends on the physical system in question as well as the context in which it is considered, since both influences the set of available measurements. For *quantum reference frames* for instance, physically distinguishable states will be implemented by *orthogonal* quantum states, *i.e.* states which can be distinguished by a quantum measurement. We will see further down why calling C the "classical reference frame states" makes sense.

Example 2.3

Consider the group $G := (\mathbb{R}, +)$ of translations in one dimension, and let $\Sigma := \mathbb{R}$ be the space of admissible positions of a particle in one dimension. Let $U(a)(b) := a + b$, *i.e.* we let translations act on the particle exactly as one would imagine. Then setting $C := \Sigma$ defines a perfect reference frame provided different positions of the particle can be distinguished with arbitrary precision.

[1]What we call "representation" of a group here is called a "realization" of a group in [32]. The concept of "representation" of [32] corresponds to our concept of *linear representation*, which will be introduced in definition 2.7 as a special case of a representation.

When explicitly constructing perfect reference frames later on, the following will be useful:

Proposition 2.4: Labelling of Classical Reference Frame States

Given a perfect reference frame Σ for G with classical reference frame states C, it is possible to label states in C as $C = \{c_g : g \in G\}$, such that

$$U(g')(c_g) = c_{g'g} \qquad \forall g, g' \in G. \tag{2.3}$$

Labellings of this type are unique up to exchanging of the labels through $g \rightsquigarrow g'g$, where $g' \in G$ is arbitrary. Equivalently, one may choose any $c \in C$ to be labelled by the identity e.

Notice in particular that the relabelling freedom means that there is nothing special about c_e, nor any other c_g. This is in line with the symmetry-breaking property, which only requires a perfect reference frame to determine the transformations which are acting, but not the initial and final states.

Proof of Proposition 2.4. From transitivity and freeness of U it follows that there is a bijection from G to C. Choosing $c_e \in C$ and imposing (2.3) then fixes the labelling of every other element. Using the representation property of U one shows that (2.3) can indeed be met for every choice of c_e, and that different choices correspond to relabelling $g \rightsquigarrow g'g$, where $c_{g'}$ is the element to be newly labelled by the identity, expressed in the old labelling system. □

Perfect Minimal Reference Frames. Note that definition 2.2 left open the possibility that Σ contained other states than the classical reference frame states C. Importantly, we have no guarantee that states outside C are any useful for breaking the symmetry of G and thus for the operation of Σ as a reference frame.

These extra states are typically not expected of reference frames as they are traditionally used in classical physics (see *e.g.* the reference frames used in special and general relativity [1, 2]). This explains why we call C the set of classical states. We introduce the following terminology:

Definition 2.5: Minimal Reference Frame

A perfect reference frame is called *minimal*, if $\Sigma = C$.

Example 2.6

Example 2.3 of a perfect reference frame is a minimal reference frame.

If we extend the example by setting $\Sigma := \mathbb{R} \times \mathbb{R}$ to be the *phase space* of the particle, *i.e.* with one factor of $\mathbb{R}$ describing position of the particle and the other its momentum, and letting G only act on the position (as one would expect), then any orbit of G in Σ can define the classical reference frame states of a perfect reference frame (assuming again perfect distinguishability). This frame will however not be minimal.

For quantum reference frames discussed below we will typically have $C \subsetneq \Sigma$, roughly speaking because quantum physics allows us to take *superpositions* of the classical states, which lie outside C. However, a quantum reference frame will *not* be the opposite of a minimal reference frame, and there exist quantum reference frames which qualify as minimal under definition 2.5. We will come back to such an example below, after having defined quantum reference frames.

2.2 Perfect Quantum Reference Frames

Recall that in pure state quantum physics [30], the state of a system is a vector $|\psi\rangle \in \mathcal{H}$ of a Hilbert space $\mathcal{H}$ up to complex multiple.[2] Explicitly, the state space is:

$$\Sigma(\mathcal{H}) := (\mathcal{H}\backslash\{0\})/_{\sim}, \qquad |\psi\rangle \sim |\varphi\rangle \Leftrightarrow \exists\, 0 \neq \lambda \in \mathbb{C} \,:\, |\psi\rangle = \lambda\, |\varphi\rangle\,.$$

Note that the zero vector is typically not considered a valid state of the system. For $0 \neq |\psi\rangle \in \mathcal{H}$ we denote by $[|\psi\rangle] \in \Sigma(\mathcal{H})$ the quantum state represented by $|\psi\rangle$. Because $\mathcal{H}$ is a Hilbert space, given a state $\sigma \in \Sigma(\mathcal{H})$ it is always possible to find a unit-norm representative $|\psi\rangle$, *i.e.* $\langle\psi|\psi\rangle = 1$ and $[|\psi\rangle] = \sigma$. As usual, two states $\sigma, \sigma' \in \Sigma(\mathcal{H})$ are *perfectly distinguishable* if any (and every) pair of representatives $|\psi\rangle$, $|\psi'\rangle$ are orthogonal:

$$|\,\langle\psi|\psi'\rangle\,|^2 = 0, \quad \text{such that} \quad [|\psi\rangle] = \sigma,\, [|\psi'\rangle] = \sigma'. \tag{2.4}$$

Probabilities are absolute squares of scalar products between unit-norm representatives of states.

Note that for a set C of states which are pairwise orthogonal one can find an observable whose possible outcomes upon measurement contains C as a subset: we simply take the one-dimensional subspaces of the states in C as the eigenspaces of this observable. Measuring that observable allows us to perfectly distinguish between the states in C, provided our system is known to be in one of those states; this justifies why we should call C a set of "perfectly distinguishable" states.

With a state space identified and meaning attributed to "perfectly distinguishable" through orthogonality of states (2.4), we could now follow definition 2.2 and define perfect quantum reference frames by setting Σ equal to $\Sigma(\mathcal{H})$. However, there are three reasons why such a definition would be inadequate; let us address them now.

1. Quantum Symmetries. To exclude pathological examples, we will assume that $U(g)$ is a *symmetry* of quantum physics $\forall\, g \in G$, *i.e.* that it leaves probabilities invariant. That is, if $\sigma_1, \sigma_2 \in \Sigma(\mathcal{H})$ are two states with representatives $|\psi_1\rangle$, $|\psi_2\rangle$, $\sigma_1' := U(g)(\sigma_1)$, $\sigma_2 := U(g)(\sigma_2)$ are their transformed counterparts with representatives $|\psi_1'\rangle$, $|\psi_2'\rangle$, then

$$\frac{|\,\langle\psi_1'|\psi_2'\rangle\,|^2}{\langle\psi_1'|\psi_1'\rangle\,\langle\psi_2'|\psi_2'\rangle} = \frac{|\,\langle\psi_1|\psi_2\rangle\,|^2}{\langle\psi_1|\psi_1\rangle\,\langle\psi_2|\psi_2\rangle}. \tag{2.5}$$

Wigner's theorem [30, 35] then guarantees that for every $g \in G$ there exists a unitary operator $\hat{U}(g) : \mathcal{H} \to \mathcal{H}$ such that for all $g \in G$ and all $\sigma \in \Sigma(\mathcal{H})$ we have

$$U(g)(\sigma) = [\hat{U}(g)\,|\psi\rangle], \qquad \forall\,|\psi\rangle \in \mathcal{H} \,:\, [|\psi\rangle] = \sigma, \tag{2.6}$$

and $g \mapsto \hat{U}(g)$ is continuous.[3] $U(g)$ can thus be seen as arising from the unitary map $|\psi\rangle \mapsto \hat{U}(g)\,|\psi\rangle$ acting on $\mathcal{H}$, but the theorem fixes $\hat{U}(g)$ only up to a g-dependent phase. The representation property (2.2) combined with (2.6) translates to

$$[\hat{U}(g')\hat{U}(g)\,|\psi\rangle] = [\hat{U}(g'g)\,|\psi\rangle], \qquad \forall\, g, g' \in G, \quad \forall\,|\psi\rangle \in \mathcal{H}, \tag{2.7}$$

which in turn means that $\hat{U}(g)$ is generally a *projective representation* of G on $\mathcal{H}$.

[2] At this point, we can equivalently identify the states of the system with the *orthogonal projectors* $\Pi(\mathcal{H})$ onto one-dimensional subspaces of $\mathcal{H}$. We will however shortly have to deal with non-normalizable states, where orthogonal projectors are ill-defined; we thus stick to the definition provided here.

[3] According to Wigner's theorem, $\hat{U}(g)$ could also be an *anti-unitary* operator. This can however be ruled out in the case of a connected Lie group, which we are considering.

Let us recall the definitions of representations on vector spaces [32, 35]:

Definition 2.7: Linear, Projective and Unitary Representations

Let V be a complex vector space.

(a) A representation $\hat{U} : G \to \mathrm{GL}(V)$ is called a *linear representation* of G.

(b) A representation $\hat{U} : G \to \mathrm{U}(V)$ is called a *unitary representation* of G. In order to distinguish it from case (c) below, we sometimes specify that $\hat{U}$ is *non-projective*.

(c) If the action $\hat{U} : G \to \mathrm{U}(V)$ is not a representation but satisfies

$$\hat{U}(g')\hat{U}(g) = \mathrm{e}^{\mathrm{i}\omega(g',g)}\hat{U}(g'g), \qquad \forall\, g, g' \in G, \tag{2.8}$$

where $\omega : G \times G \to \mathbb{R}$ is a continuous function, we say that $\hat{U}$ is a *projective representation* of G.

Without specification, a "unitary" representation is always taken to be non-projective, *i.e.* of type (b). We will always explicitly state if we consider a projective representation.

Note that linear and unitary representations are representations in the sense of definition 2.1, but projective representations are not. Also, one can show that ω satisfies a so-called *2-cocycle equation* [36]

$$\omega(g, g') + \omega(gg', g'') = \omega(g', g'') + \omega(g, g'g''), \qquad \forall\, g, g', g'' \in G, \tag{2.9}$$

and

$$\omega(g, e) = \omega(e, g), \qquad \forall\, g \in G. \tag{2.10}$$

2. Improper States. Since G is a Lie group, it is uncountable, implying that C must be uncountable too. We thus have the task of finding an uncountable set of mutually orthogonal states in $\mathcal{H}$. This is a problem, since Hilbert spaces can have at most countable bases of mutually orthogonal states. We can however obtain a perfect reference frame if we also allow *improper states*, as illustrated by the following example:

Example 2.8

Consider a quantum particle in one dimension, described by $\mathcal{H} := L^2(\mathbb{R})$, and take again $G := (\mathbb{R}, +)$ to be the translation group in one dimension. Take the standard translation of the wave function $\psi \in L^2(\mathbb{R})$ as a representation:

$$(\hat{U}(a)(\psi))(x) := \psi(x - a). \tag{2.11}$$

This is the quantized version of the classical system considered in example 2.3.

As such, it would seem intuitive that one can use the quantum particle to define a perfect quantum reference frame for $(\mathbb{R}, +)$. This suspicion is further strengthened by the rule of thumb that classical observables (in our case position x) become operators ($\hat{x}$ here), with the possible measurement outcomes precisely the classically allowed values of the observable ($\mathbb{R}$ for us). Thus: by taking the projectors constructed from the position eigenstates $|y\rangle$, $y \in \mathbb{R}$ and $\hat{x}\,|y\rangle = y\,|y\rangle$, we should be able to construct a set C of classical reference frame states.

This fails because the position eigenstates are *improper states* not contained in $\mathcal{H}$, but rather in a larger, non-Hilbert vector space of distributions. If the L^2-norm $\|\cdot\|$ of $\mathcal{H}$ is extended to that space, the norms of position eigenstates are infinite. In this case, the larger space is the space $\mathcal{S}^\times(\mathbb{R})$ of *tempered distributions*. This space includes Dirac-δ distributions (which are the position eigenstates), but also other non-normalizable states such as the plane waves representing *momentum eigenstates*.

Issues such as this can be resolved by switching from the Hilbert space formalism to so-called *Rigged Hilbert spaces* [30, 31]. Roughly speaking, one generally proceeds as in the example above: we identify a larger vector space containing our Hilbert space and additional, new states of interest; the scalar product is then partially extended to this larger space. The larger space will typically fail to be Hilbert space, often containing pairs of states with infinite scalar products. The extended scalar product will however be enough to give a useful meaning to Hermitian transposition, unitarity, *etc.* on the larger space [37]. Also, given a linear map between Hilbert spaces it is under quite general assumptions possible to extend it uniquely and continuously to a linear map between the corresponding larger spaces [37]; this in particular allows extensions of representations [38]. One can then effectively work with the larger space instead of the Hilbert space.

We will not need the details of the rigged Hilbert space formalism and refer to the literature. We will deal with the problem slightly informally (as is often done) by treating the extended space as if it were a Hilbert space, knowing that the details can be worked out in the context of rigged Hilbert spaces. For simplicity, we will not adopt the standard notation of rigged Hilbert spaces[4] and write $\bar{\mathcal{H}}$ for the larger space given a Hilbert space $\mathcal{H}$. Note that the precise meaning of the larger space depends on the situation, but again we will not need the details.

To use $\bar{\mathcal{H}}$ in quantum reference frames, we must construct Σ from $\bar{\mathcal{H}}$:

$$\Sigma := \Sigma(\bar{\mathcal{H}}), \tag{2.12}$$

On $\Sigma(\mathcal{H}) \subset \Sigma(\bar{\mathcal{H}})$ it is possible to find representatives with unit norm, but in $\bar{\mathcal{H}}$ this is generally not possible. Because of this, the notion of quantum symmetry described above, which relies on preservation of probabilities in turn relying on normalized states, must be adjusted: we will instead assume that the representation U on Σ stems from a projective representation $\hat{U}$ on $\mathcal{H}$ as in Wigner's theorem, extended to $\bar{\mathcal{H}}$. Restricted to $\Sigma(\mathcal{H})$, this then implies conservation of probabilities.

3. Projective Representations and Central Extensions. It is mathematically possible to see a projective representation of G as a unitary, *non-projective* representation of a larger group $\mathrm{C}G$, a *central extension of* G [36]. This is why one can for instance study spin using unitary representations of SU(2) instead of projective representations of SO(3). Furthermore, in the case of the Galilei group, it can even be argued that the physically relevant group is the central extension [39]. Thus, *we will from now on focus only on unitary, non-projective representations*, and we will not mention that a representation is non-projective.

[4] Roughly speaking, if $\mathcal{H}$ is a Hilbert space and $\Phi \subset \mathcal{H}$ is a suitable subspace of "test functions", and $\Phi^\times$ is the corresponding "distribution space" (technically, complex-conjugated distribution space), then the triple $\Phi \subset \mathcal{H} \subset \Phi^\times$ defines a rigged Hilbert space. The "larger space" is $\Phi^\times$. See [30] for an introduction to Rigged Hilbert spaces and [31] for a detailed treatment.

Definition of Perfect Quantum Reference Frames. With all three of these points cleared, we can finally proceed with defining *perfect quantum reference frames*:

Definition 2.9: Perfect Quantum Reference Frame

Consider a quantum system described by a rigged Hilbert space $\mathcal{H} \subset \bar{\mathcal{H}}$ with state space $\Sigma = \Sigma(\bar{\mathcal{H}})$. Let $\hat{U}$ be a unitary representation of G on $\bar{\mathcal{H}}$,[5] giving rise to a representation U on Σ.

If there exists a subset $C \subset \Sigma$ satisfying the three requirements of definition 2.2, in particular with perfect distinguishability defined via orthogonality (2.4), then we call the system a *perfect quantum reference frame for G* with classical reference frame states C.

As mentioned before, a quantum reference frame is usually not minimal, since it is often possible to find superopositions of states in C, which are themselves not in C.

On the level of Hilbert spaces we note the following useful characterization of distinguishability of the classical states:

Proposition 2.10

For a perfect quantum reference frame it holds that for every $e \neq g \in G$ and all $c \in C$

$$\langle\Gamma| \hat{U}(g) |\Gamma\rangle = 0, \qquad \forall\ |\Gamma\rangle \in \bar{\mathcal{H}}\ :\ [|\Gamma\rangle] = c. \tag{2.13}$$

Proof. Since G acts freely on C, for $e \neq g \in G$, $c \in C$ and any representative $|\Gamma\rangle$ of c it holds that $U(g)(c) = [\hat{U}(g) |\Gamma\rangle] \neq c$. Thus, $U(g)(c)$ must be perfectly distinguishable from c, which according to (2.4) means that $\langle\Gamma| \hat{U}(g) |\Gamma\rangle = 0$. □

Also, we introduce *representatives of C*:

Definition 2.11

For a perfect quantum reference frame with classical states C, for every $|\Gamma\rangle \in \bar{\mathcal{H}}$ representing a state $c \in C$, we introduce the set

$$C_{|\Gamma\rangle} := \left\{\hat{U}(g) |\Gamma\rangle\ :\ g \in G\right\} \subset \bar{\mathcal{H}}. \tag{2.14}$$

It holds that $[C_{|\Gamma\rangle}] = C$, making $C_{|\Gamma\rangle}$ a *set of representatives of the classical states.* This set is minimal and also an orbit of $\hat{U}$. We will often work with some $C_{|\Gamma\rangle}$ instead of C. The different $C_{|\Gamma\rangle}$ differ by a multiplicative constant stemming from normalization.

Finally, we note that one can more generally allow *mixed* reference frame states as opposed to *pure* ones described so far. We will do this, but keep the classical reference frame states pure for simplicity. The switch to mixed states works rigorously when restricting to $\mathcal{H}$ by considering *density operators on $\mathcal{H}$*; see *e.g.* [30] for a treatment of mixed state quantum physics. However, we will also have to deal with mixtures of non-normalizable states. For those we do not seek a rigorous treatment but instead content ourselves with formal expressions and keep track of infinities. For instance, we will extend the trace to non-normalizable orthogonal bases in some places; we will come back to this once required.

Note that what we call a "perfect quantum reference frame" here corresponds to an "ideal reference frame" (a special type of "complete reference frame") in [13].

[5] More precisely, we consider $\hat{U}$ to be unitary on $\mathcal{H}$, with extension to $\bar{\mathcal{H}}$.

2.3 Perfect Quantum Reference Frames from Regular Representations

We saw in example 2.8 how the square-integrable functions $L^2(\mathbb{R})$ with representation (2.11), augmented by δ-distributions, could be used as a perfect quantum reference frame for $(\mathbb{R}, +)$. Note that we can understand (2.11) as $a \in \mathbb{R}$ acting on the argument x of a function ψ by addition (the group multiplication in $(\mathbb{R}, +)$) of $-a$ (the group inverse of a in $(\mathbb{R}, +)$). A similar construction will allow us to define a perfect quantum reference frame for any Lie group G using the square-integrable functions on G, see *e.g.* [13]. Let us make this precise.

To define integration on G, one needs a *measure* on G. Measures are conveniently provided by the left and right *Haar measures* [34] which exist for every Lie group G. These measures have the property of being left- and right-invariant respectively under the action of group multiplication in G, similarly to how the *Lebesgue measure* used to define $L^2(\mathbb{R})$ is invariant under translations, *i.e.* under group multiplication in $(\mathbb{R}, +)$; in fact, the Lebesgue measure is a left *and* right Haar measure of $(\mathbb{R}, +)$. Both left and right Haar measures are unique up to a positive multiplicative constant. If left and right Haar measures coincide, we call G *unimodular* [34]. The basics of Haar measures are treated in appendix B.1.

While not all Lie groups are unimodular, most Lie groups used in physics are. The following are some of the most often-encountered unimodular Lie groups in physics [34, 40]:

(a) All *compact* Lie groups are unimodular. This includes the orthogonal or unitary groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$, as well as their special variants $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$.

(b) All *Abelian* Lie groups are unimodular. This includes the *translation groups* in $\mathbb{R}^n$, of which the *Galilei group in one dimension* is a special case.

(c) The general and special linear groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ are unimodular.

(d) The *strictly upper triangular matrices* with unit diagonal in $\mathrm{GL}(n)$ are unimodular. In particular, the *Heisenberg group* is unimodular.

(e) The *Lorentz* and *symplectic* groups are unimodular.

As for *non-unimodular* Lie groups in physics, one finds *affine groups* [40] as the only examples which are reasonably well-known. Of course, the above list is not exhaustive; it is however extensive enough so that we do not lose much generality when assuming G to be unimodular. Since it is instructive to see the differences or rather analogies in properties of left- and right Haar measures when introducing them, we will in this section *not* assume unimodularity. Starting from the next section however, *G will always be unimodular.*

With left and right Haar measures at hand we can define two notions of square-integrable functions [41]:

Definition 2.12: Square-Integrable Functions on G

Let μ_L and μ_R be left and right Haar measures of G. Use the measures to define two notions of integration and two scalar products between complex functions $\psi, \varphi : G \to \mathbb{C}$ as

$$\langle\psi|\varphi\rangle := \int_G \mathrm{d}\mu(g)\, \psi^*(g)\varphi(g), \qquad \mu = \mu_L \text{ or } \mu = \mu_R \text{ respectively.} \tag{2.15}$$

These two scalar products define the vector spaces $L^2(G, \mu_L)$ and $L^2(G, \mu_R)$ of square-integrable, complex-valued functions on G. For simplicity, we often write $\mathrm{d}g = \mathrm{d}\mu_L(g)$. If G is unimodular we always choose $\mu_L = \mu_R$, so that both spaces coincide, and we can unambiguously write $L^2(G)$.

We will often use the bra-ket notation and write $|\phi\rangle = \phi$.

One can show that $L^2(G, \mu_L)$ and $L^2(G, \mu_R)$ are Hilbert spaces [42].

Due to μ_R not necessarily being left-invariant, we can only generalize the representation (2.11) to $L^2(G, \mu_L)$, where we obtain the unitary *left-regular representation*. One can however introduce the unitary *right-regular representation* on $L^2(G, \mu_R)$, which corresponds to $\psi(x) \mapsto \psi(x+a)$ in the context of example 2.8. More precisely [41] (for details, see also appendix B.2):

Definition 2.13: Left- and Right-Regular Representations

The *left-regular representation* $\hat{L}$ on $\psi \in L^2(G, \mu_L)$ is the unitary representation

$$(\hat{L}(g)\psi)(g') := \psi(g^{-1}g'), \qquad \forall\, g, g' \in G, \quad \forall\, \psi \in L^2(G, \mu_L). \tag{2.16}$$

The *right-regular representation* $\hat{R}$ on $L^2(G, \mu_R)$ is the unitary representation

$$(\hat{R}(g)\psi)(g') := \psi(g'g), \qquad \forall\, g, g' \in G, \quad \forall\, \psi \in L^2(G, \mu_R). \tag{2.17}$$

In the unimodular case, we quickly see that:

Proposition 2.14: Left- and Right-Regular Representations Commute

Let G be unimodular, *i.e.* both $\hat{L}$ and $\hat{R}$ act on $L^2(G)$. Then:

$$\big[\hat{R}(g), \hat{L}(g')\big] = 0, \qquad \forall\, g, g' \in G. \tag{2.18}$$

Proof. For $g, g' \in G$, $\psi \in L^2(G)$, we find

$$\begin{aligned}\hat{R}(g)\hat{L}(g')\psi &= \hat{R}(g)\big(g'' \mapsto \psi(g'^{-1}g'')\big)\\ &= g'' \mapsto \psi(g'^{-1}g''g) = \hat{L}(g')\big(g'' \mapsto \psi(g''g)\big) = \hat{L}(g')\hat{R}(g)\psi.\end{aligned} \tag{2.19}$$

□

Analogously to δ-distributions on $\mathbb{R}$ [43], we define δ_g as the δ-distribution on G with peak at $g \in G$ [44]:

Definition 2.15: δ-Distributions on G

For $g \in G$, the action of the distribution δ_g on a test function $\psi : G \to \mathbb{C}$ is set to be

$$\delta_g[\psi] := \int_G \mathrm{d}\mu(g')\, \delta_g(g')\psi(g') := \psi(g), \tag{2.20}$$

where $\mu = \mu_L$ or $\mu = \mu_R$ depending on the context. We abbreviate $\delta := \delta_e$. In bra-ket notation we write $|g\rangle := \delta_g$ and $\langle g|\psi\rangle = \langle\psi|g\rangle^* := \delta_g[\psi] = \psi(g)$.

The left- and right-regular representations are extended to δ-distributions as

$$\hat{L}(g)\,|g'\rangle := |gg'\rangle\,, \tag{2.21}$$

$$\hat{R}(g)\,|g'\rangle := \left|g'g^{-1}\right\rangle. \tag{2.22}$$

The extension of the bra-ket notation to these distributions is analogous to how we work with position states $|x\rangle$ in the context of $L^2(\mathbb{R})$, see *e.g.* [30]. The extension of $\hat{L}$ and $\hat{R}$

results in “scalar products” of the form $\langle g|\psi\rangle$ being conserved under $\hat{L}$ and $\hat{R}$.

The integral in (2.20) suggests that we can almost think of the δ_g's as functions; and since this works in the context of $L^2(\mathbb{R})$ (see *e.g.* [30]), we expect it to work here too. As shown in proposition B.5 of appendix B.2, this is indeed the case. An important part of this result is that we can extend scalar products to the δ_g's as if they were functions.

We mention here some results in the unimodular case $L^2(G)$ (see proposition B.5 for the more general case). Firstly, the extension of the scalar product yields

$$\langle g'|g\rangle = \langle g|g'\rangle = \delta(g'^{-1}g) = \delta(g^{-1}g') = \delta(gg'^{-1}) = \delta(g'g^{-1}). \tag{2.23}$$

Notably, $\langle g'|g\rangle = 0$ if $g' \neq g$. Secondly, the completeness relation holds:

$$\int_G \mathrm{d}g \; |g\rangle\langle g| = \hat{\mathrm{id}}. \tag{2.24}$$

Thirdly, it is possible to expand square-integrable functions $\psi \in L^2(G)$ as

$$|\psi\rangle = \int_G \mathrm{d}g \, \psi(g) \, |g\rangle \,, \tag{2.25}$$

similarly to the well-known expansion of functions in $L^2(\mathbb{R})$ in terms of position states $|x\rangle$.

One can think of other distributions on G which can be thought of as functions: for instance, any non-square integrable function, such as *plane waves* $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ or *Heaviside-step functions* on $(\mathbb{R}^n, +)$; another set of examples are various linear combinations of δ-distributions, including infinite ones such as *Dirac-combs* on $\mathbb{R}$. One can formally extend the scalar product (2.15) for all these functions in an analogous way as for δ-distributions.

With all this preparation it is now easy to see the main result of this section:

Theorem 2.16: Perfect Quantum Frames from Regular Representations

$\mathcal{H} := L^2(G, \mu_L)$ with $\hat{L}$ and $C_{|e\rangle} := \{\hat{L}(g)\,|e\rangle \;:\; g \in G\}$ is a perfect quantum reference frame for G.

The same is true for $\mathcal{H} := L^2(G, \mu_R)$ with $\hat{R}$ and $C_{|e\rangle} := \{\hat{R}(g)\,|e\rangle \;:\; g \in G\}$.

The set $C_{|e\rangle}$ of classical state representatives is the same in either case.

These reference frames are widely used in the literature (see *e.g.* reference frame sources in the introduction). From now on, when speaking of perfect quantum reference frames, we always mean those. Figure 2.1 illustrates how a clock hand whose angle states are orthogonal can be seen as $L^2(\mathrm{U}(1))$, a perfect reference frame for the group of rotations around a single axis.

Proof of theorem 2.16. By definition, $C_{|e\rangle}$ is an orbit of $\hat{L}$ and $\hat{R}$; in both cases it is equal to $\{|g\rangle \;:\; g \in G\}$. Since there is a bijection between G and $C_{|e\rangle}$, the action of $\hat{L}$ ($\hat{R}$) must be free; alternatively one may check directly that $\hat{L}(g)$ acting on $|g'\rangle$ produces a different state for every $g \in G$, and similarly for $\hat{R}(g)$. The orthogonality (B.8) implies that the states in $C_{|e\rangle}$ are perfectly distinguishable. □

2.4 Imperfect Reference Frames

An *imperfect reference frame* is a reference frame whose classical states do not perfectly break the symmetry of G. This can happen in two ways: the classical states C are not

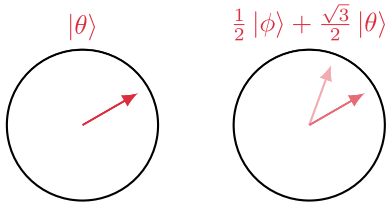


Figure 2.1: Left: The perfect reference frame $L^2(\mathrm{U}(1))$ for the group $\mathrm{U}(1)$ of rotations around a single axis can be thought of as the Hilbert space of a clock hand, with mutually orthogonal definite-angle states $|\theta\rangle$, $\theta \in [0, 2\pi)$, *i.e.* $\mathrm{e}^{\mathrm{i}\theta} \in S^1 \cong \mathrm{U}(1)$, as classical states.

Right: Quantum reference frames allow for superpositions of classical reference frame states. Typically, not every reference frame state is a classical reference frame state.

The group $\mathrm{U}(1)$ plays an important role whenever phases are involved. Quantum reference frames for $\mathrm{U}(1)$ were thus among the first to be studied [3, 4], but also more recently [23].

perfectly distinguishable, and/or the action of G could be non-free. Note that non-freeness immediately implies imperfect distinguishability, since some classical reference frame states will be identical. That is, we only demand property (b) in definition 2.2. Also, we will exclude perfect reference frames from being special cases of imperfect reference frames.

The same logic defines *imperfect quantum reference frames*: the representation of G could be non-free, and/or the classical reference frame state could be pairwise non-orthogonal. We however would still like to be able to measure imperfect quantum reference frames, so we require that the classical reference frame state loosely speaking form a *positive operator-valued measure* (POVM) [45].[6]

We thus define:

Definition 2.17: Imperfect (Quantum) Reference Frame

Consider a physical system with state space Σ, acted upon by a representation U of G. If there is a set of states $C \subset \Sigma$ such that

(a) C is an orbit of G under U (*i.e.* G acts transitively on C),

(b) Σ, U and C do not form a perfect reference frame,

then the system is called an *imperfect reference frame for G* with classical reference frame states C.

Consider a quantum system described by the rigged Hilbert space $\mathcal{H} \subset \bar{\mathcal{H}}$, $\Sigma = \Sigma(\bar{\mathcal{H}})$, acted upon by a unitary representation $\hat{U}$ of G giving rise to a representation U on Σ. If there exists a subset $C \subset \Sigma$ such that it satisfies the conditions (a) and (b) above and for any representative $C_{|\Gamma\rangle}$ of C we have that[7]

$$\exists\, r > 0 \quad \forall\, |\varphi\rangle \in C_{|\Gamma\rangle} \quad : \quad \frac{1}{r}\int_G \mathrm{d}g\, \hat{U}(g)\, |\varphi\rangle\langle\varphi|\, \hat{U}^\dagger(g) = \hat{\mathrm{id}}, \tag{2.26}$$

[6]It would be sufficient to require that the classical states are part of a POVM; this would be a case where measuring the imperfect reference frame can also result in other states (say corresponding to a "failed" experiment) besides classical reference frame states. But we can exclude this possibility without loss of generality by restricting our Hilbert space to the span of the orbit of classical states. For perfect reference frames, the classical reference frame states are orthogonal and thus always part of a POVM.

then we call the quantum system an *imperfect quantum reference frame for* G with classical reference frame states C.

We will mostly be interested in imperfect quantum reference frames. Roughly speaking, a quantum reference frame is imperfect, if $\dim \operatorname{span} C_{|\Gamma\rangle} < \#G$, where $\#G$ is the number of elements in G. Our imperfect quantum reference frames correspond to the "incomplete" and/or "non-ideal" reference frames studied in [13].

Note that if G is compact, then the integral (2.26) always converges. To see this, we consider any matrix element $\langle\psi_1| \int_G \mathrm{d}g\, \hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g) |\psi_2\rangle = \int_G \mathrm{d}g\, \langle\psi_1| \hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g) |\psi_2\rangle$, which is the integral of a continuous function on G, and thus converges because G is compact. This is true for all matrix elements, hence the integral in (2.26) is well-defined. If G is non-compact, then this may no longer be the case.

Checking for completeness (2.26) is easy if $\hat{U}$ is an *irreducible* representation, thanks to *Schur's lemma* [30]: *any operator which commutes with the irreducible representation must be a multiple of the identity.* In our case, $\int_G \mathrm{d}g\, \hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g)$, assuming it is well-defined, commutes with the representation:

$$\begin{aligned}\int_G \mathrm{d}g\, \hat{U}(g')\hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g) &= \int_G \mathrm{d}g\, \hat{U}(g'g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g) \\ &= \int_G \mathrm{d}g\, \hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g'^{-1}g) = \int_G \mathrm{d}g\, \hat{U}(g) |\varphi\rangle\langle\varphi| \hat{U}^\dagger(g)\hat{U}(g'). \end{aligned} \quad (2.27)$$

Thus, if $\hat{U}$ is irreducible and as long as the left-hand side of (2.26) is well-defined, the completeness relation (2.26) is satisfied. We will state a more general version of this result further down.

A labelling of classical states like in proposition 2.4 is still possible, even when the representation of G is non-free, if we allow for classical states to be labelled more than once:

Proposition 2.18: Labelling of Classical Reference Frame states

Given the classical states C of an imperfect quantum reference frame one can consider the representative $C_{|\Gamma\rangle} = \{|\Gamma_g\rangle \,:\, g \in G\}$, with

$$\hat{U}(g') |\Gamma_g\rangle = |\Gamma_{g'g}\rangle \,, \qquad \forall\, g, g' \in G, \quad (2.28)$$

allowing for each vector in $C_{|\Gamma\rangle}$ to be labelled multiple times if the representation $\hat{U}$ is non-free.

The same argument used in the proof of 2.4 works here too. An orbit $\{|\Gamma_g\rangle\}_{g\in G}$ of states satisfying a completeness relation (2.26) are sometimes called a *set of coherent states* [46]; we will however not use this terminology and reserve "coherent" for so-called *squeezed coherent states* of quantum particles in chapter 6.

[7] We will see further down that it makes sense to allow formally infinite constants $r > 0$.

We mentioned earlier that quantum reference frames can be minimal too. This is for instance possible with imperfect quantum reference frames:

Example 2.19

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and $G = \mathrm{SU}(\mathcal{H})$ its *special unitary group* acting in the defining representation on $\mathcal{H}$. The orbit of any state $\sigma \in \Sigma$ under the defining representation is the entire state space Σ, because for any two unit-norm vectors $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$, there exists a unitary operator $\hat{U}$ such that $|\psi\rangle = \hat{U}|\varphi\rangle$. This also shows that the defining representation of $\mathrm{SU}(\mathcal{H})$ is irreducible.

To build a reference frame with $\mathcal{H}$ and the defining representation of $\mathrm{SU}(\mathcal{H})$, one is thus forced to choose $C = \Sigma$ and every potential quantum reference frame is minimal. Also, this reference frame would be imperfect, if $\dim \mathcal{H} < \#\mathrm{SU}(\mathcal{H})$. This is the case for $\dim \mathcal{H} > 1$. The case $\dim \mathcal{H} = 1$ is trivial, since then $\mathrm{SU}(\mathcal{H}) = \{\hat{\mathrm{id}}\}$. For concreteness, we can take $\mathcal{H} = \mathbb{C}^2$ and thus $G = \mathrm{SU}(2)$. One checks: $\dim \mathcal{H} = 2 < \#\mathrm{SU}(2) = \infty$.

For $\dim \mathcal{H} > 1$ it remains to check the completeness relation (2.26), to see whether $\mathcal{H}$ with the defining representation of $\mathrm{SU}(\mathcal{H})$ is an actual imperfect reference frame. The integral converges, because $\mathrm{SU}(\mathcal{H})$ is compact, and the relation holds, because the defining representation is irreducible.

Formal Infinities If G is non-compact, then the integral on the left-hand side of (2.26) is not always well-defined, as the following example illustrates:

Example 2.20

Take $G := (\mathbb{R}^2, +)$, the group of vector addition in the plane, $\mathcal{H} := L^2(\mathbb{R})$, and for $(a, b) \in \mathbb{R}^2$, $\phi \in L^2(\mathbb{R})$, define the representation $(\hat{U}(a,b)(\phi))(x) := \mathrm{e}^{\mathrm{i}\varphi(b)}\phi(x-a)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\varphi(b) + \varphi(b') = \varphi(b+b')$. In other words, our states are sensitive only to the translation component along one axis, while the component along the other axis provides merely a phase. One checks that this can be made to satisfy (a) and (b) in definition 2.17, *e.g.* by taking position eigenstates as classical states.

For it to truly become an imperfect quantum reference frame, the completeness (2.26) must be satisfied. we have

$$\int_G \mathrm{d}g\, \hat{U}(g)\, |x\rangle\langle x|\, \hat{U}^\dagger(g) = \int_{\mathbb{R}} \mathrm{d}a\, \mathrm{d}b\, |x-a\rangle\langle x-a|\,. \tag{2.29}$$

With no need for further manipulations we can see already now that $\int \mathrm{d}b$ will introduce a factor of ∞, surely rendering the left-hand side ill-defined.

We could solve this problem by restricting the integral $\int_G$ to the subgroup of translations along the first axis. This would also make sense physically, since translations along the other axis could in any case not be measured, removing the need for the POVM to include those transformations. An equivalent way to deal with the problem is to leave the integral as is, but to *formally* normalize by the infinite constant introduced through $\int_{\mathbb{R}} \mathrm{d}b$. We choose the latter option.

Naively, this example seems rather contrived. It however serves to illustrate an important more general case: when our group factors topologically as $G \cong G' \times X$, with $G' \subset G$ an *unbounded* subgroup and X a topological space, such that $\mu = \mu' \cdot \mu_X$, where μ' is a

Haar measure on G' and μ_X a measure on X, and G' acts through $\hat{U}$ only as a phase (in particular, if it does not act at all) then the integral is ill-defined. This can happen only if G itself is non-compact.[8] This precisely happens for the centrally extended Galilei group (see section 5.3): besides translations and boosts, one also has "θ-translations" needed to make the representation unitary; these latter translations take values in $\mathbb{R}$, which is unbounded, and act only through a phase factor on our Hilbert space. More generally, we can expect divergences when considering central extensions of groups by a non-compact Abelian group, such as $(\mathbb{R}, +)$ in the case of the Galilei group.[9] Finally, it is conceivable that the integral diverges also in some cases where G is non-compact, but there is no clearly identifiable subgroup which acts only as a phase and hence introduces an infinity.

To deal with situations such as this, we loosen the definition 2.17 to allow for *formally infinite* proportionality constants in (2.26); equivalently, the left-hand side of (2.26) must be a well-defined operator up to possibly a formally infinite constant. We will not be able to explain these infinities rigorously using rigged Hilbert spaces, as they do not originate in non-normalizable states, but rather in non-compact Lie groups. We will adopt the strategy of treating these infinities as *symbols* for which arithmetic operations are partially defined, and by meticulously keeping track of them. In particular, we allow addition, subtraction, multiplication, division, and more general powers of these infinite symbols, enabling especially the "cancelling" of symbols. Importantly, one formal infinity can only be cancelled by itself, and not by other formal infinities. Also, we cannot define expressions of the type "$0 \cdot \infty$" in general. One must investigate every case where these infinities occur separately. This approach will be enough for us. The manipulation of formal infinities can however be made precise, and we give a rigorous description in terms of *field extensions* of $\mathbb{C}$ in appendix B.3.

In conclusion, we replace the real constant $r > 0$ in (2.26) with a possibly formally infinite positive constant. If the infinite factor was a result of an unbounded subgroup acting only as a phase, then its introduction into our calculations should not interfere with any physical aspects, since the unbounded subgroup has no real physical meaning. In particular, we still expect to be able to compute probabilities correctly. The case of the centrally extended Galilei group relevant for us luckily falls into this category.

Irreducible Representations. With formal infinities in place, let us come back to the special case of an irreducible representation. We can now more generally state:

Proposition 2.21: Imperfect Frames in Irreducible Representations

If $\hat{U}$ on $\mathcal{H}$, then the orbit of every $|\psi\rangle \in \bar{\mathcal{H}}$, for which $\frac{1}{r}\int_G \mathrm{d}g\, \hat{U}(g)\, |\psi\rangle\langle\psi|\, \hat{U}^\dagger(g)$ is a well-define operator with $r > 0$ a potentially formally infinite constant, can be used as classical reference frame states in order to define an imperfect reference frame.

This is simply a relaxation of the above statement, allowing for formally infinite constants.

[8] Unbounded G' means $\mu'(G') = \infty$, which according to theorem B.4 is the case if and only if G' is non-compact. But $G' \cong G' \times \{x\}$ for some $x \in X$ is closed in G, and thus can only be non-compact if G is non-compact.

[9] It is also possible to extend using a compact Abelian group [36], which could resolve some issues with infinities. We briefly consider this option for the Galilei group in section 5.3; but since it leads to an unnaturally quantized mass, we will not pursue this approach.

A weak version of the converse also holds:

Proposition 2.22: Sufficient Condition for Irreducibility

If $0 \neq \frac{1}{r} \int_G \mathrm{d}g\, \hat{U}(g) |\psi\rangle\langle\psi| \hat{U}^\dagger(g) \propto \hat{\mathrm{id}}$ holds for all $0 \neq |\psi\rangle \in \mathcal{H}$ and is a well-defined operator for some potentially formally infinite constant $r > 0$, then $\hat{U}$ is irreducible.

Proof. Assume towards contradiction that $\mathcal{H}$ has an invariant, true subspace $\{0\} \subsetneq \mathcal{V} \subsetneq \mathcal{H}$. Take then $0 \neq |\psi\rangle \in \mathcal{V}$ and $0 \neq |\varphi\rangle \in \mathcal{V}^\perp$. Because $\mathcal{V}$ is invariant, $\hat{U}(g)|\psi\rangle \in \mathcal{V}$, $\forall\, g \in G$, and hence $\langle\psi| \hat{U}^\dagger(g) |\varphi\rangle = 0$, $\forall\, g \in G$. Thus, $\frac{1}{r} \int_G \mathrm{d}g\, \hat{U}(g) |\psi\rangle\langle\psi| \hat{U}^\dagger(g)$ has the eigenvalue zero and cannot be a non-zero multiple of the identity, contradicting our assumptions. □

This result will be useful later to prove that certain representations of the Galilei group are irreducible. Note that one must check *every* state $|\psi\rangle \in \mathcal{H}$ at least up to scaling. Note also that in these propositions, $\frac{1}{r} \int_G \mathrm{d}g\, \hat{U}(g) |\psi\rangle\langle\psi| \hat{U}^\dagger(g)$ can be well-defined while $\int_G \mathrm{d}g\, \hat{U}(g) |\psi\rangle\langle\psi| \hat{U}^\dagger(g)$ could diverge and thus not be well-defined.

2.5 Badness of Imperfect Quantum Reference Frames

Let us briefly turn to the task of assessing the usefulness of different imperfect quantum reference frames. This will particularly be practical when dealing with explicit examples of imperfect reference frames of the Galilei group in chapter 6. The assessment of "goodness" or "badness" of imperfect reference frames can be approached in various ways and different approaches can be useful depending on the circumstances considered; see *e.g.* [19] and [47] for examples of approaches. We introduce here our own.

We will always work with a representative $C_{|\Gamma\rangle}$ of classical states and adopt a labelling like in proposition 2.18.

Badness from Orthogonality of Classical States. Roughly speaking, we want to term an imperfect reference frame as "worse" than another, if the classical states $C_{|\Gamma\rangle}$ of the former have an overall larger overlap with each other, *i.e.* have higher *fidelity* [45], and are thus less distinguishable than the classical states of the latter. So up to normalization, expressions of the type

$$\big|\, \langle\Gamma_{g'}|\Gamma_g\rangle \,\big| \tag{2.30}$$

should contribute to badness. A specific form of badness would then be to cumulate all possible such overlaps; *i.e.* it would be the integral of (some function of) these overlaps over g and g'. Equivalently, we may integrate (some function of)

$$\big|\, \langle\Gamma_e| \hat{U}(g'^{-1}gg') |\Gamma_e\rangle \,\big| \tag{2.31}$$

over g and g'. Because G is assumed to be unimodular, the Haar measure is conjugation-invariant, and we can substitute $g \rightsquigarrow g'gg'^{-1}$, rendering the integral over g' pointless.

Accordingly, we define:

Definition 2.23: Badness Measure of Imperfect Reference Frame

Given an imperfect reference frame $C_{|\Gamma\rangle}$ of G, a *badness measure* $B(C_{|\Gamma\rangle})$ is an expression of the form

$$B(C_{|\Gamma\rangle}) = \int_G \mathrm{d}g\, f\big(|\, \langle\Gamma_e| \hat{U}(g) |\Gamma_e\rangle \,|\big), \tag{2.32}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function.

Importantly, we can replace $|\Gamma_e\rangle$ by any other $|\Gamma_g\rangle$. This is important, since the badness measure should not depend on the arbitrary choice of which classical state is labelled by the identity.

Note that the "figures of merit" of [19] resemble these badness measures. They consider the situation where classical reference frame states are used in a communication setting to send information and states of reference frames are measured by a POVM which does not necessarily correspond to the set of classical reference frame states; they go on to show that it is sensible to align the POVM with the classical states.

A Naive Badness Measure. With the above considerations, we can define a simple badness measure:

$$B(C_{|\Gamma\rangle}) := \frac{1}{|\langle\Gamma_e|\Gamma_e\rangle|^2} \int_G \mathrm{d}g \left| \langle\Gamma_e| \hat{U}(g) |\Gamma_e\rangle \right|^2. \tag{2.33}$$

The prefactor is chosen in order to yield a result independent of normalization of $|\Gamma_e\rangle$. Due to this normalization, the measure is even independent of representative and thus truly only depends on the reference frame C; we can unambiguously write $B(C) := B(C_{|\Gamma\rangle})$. But (2.26) implies that this measure is in fact *uniform*, and thus it is not at all useful.

It is interesting to note that [47] use a badness measure like (2.33), but instead of the classical reference frame state $|\Gamma_e\rangle$ they use the state of a system observed in the reference frame in question. This state can still be made into an orbit under G by transforming it to other reference frame orientations, but it does not necessarily satisfy a completeness relation, rendering the badness measure more useful.

Weighted Badness Measures. We can make (2.33) more interesting if we introduce a *weight* $\Omega_0(g) \geq 0$. The weight allows us to specify which kinds of transformations $g \in G$ we deem more important than others. As an example which will be relevant later (for details, see section 6.3), consider an imperfect reference frame for the *Galilei group in one dimension* (essentially translations and boosts in one dimension); in a given context we might not be able to measure position and velocity of particles with equal precision and so it would make sense to require our imperfect frame to be more accurate for one of the two types of Galilei transformations. Thus, a good badness measure would be one which weights one type of transformation more than the other, as shown in figure 2.2.

Let us return to the general case. Naively, the weight would be introduced as a factor in the integral (2.33), as

$$\frac{1}{|\langle\Gamma_e|\Gamma_e\rangle|^2} \int_G \mathrm{d}g\, \Omega_0(g) \left| \langle\Gamma_e| \hat{U}(g) |\Gamma_e\rangle \right|^2. \tag{2.34}$$

But when we do this, we find that in order for the expression to be invariant under $|\Gamma_e\rangle \rightsquigarrow \hat{U}(g') |\Gamma_e\rangle = |\Gamma_{g'}\rangle$, $g' \in G$, we would need Ω_0 to be *conjugation-invariant*; that is, $\Omega_0(g'^{-1}gg') = \Omega_0(g), \forall\, g, g' \in G$. While our badness measure should be invariant under $|\Gamma_e\rangle \rightsquigarrow |\Gamma_{g'}\rangle$, we also want the freedom to truly weigh every group element as we please.

The reason for this unexpected result is that equation (2.33) is not the right place to introduce Ω_0. What we really want is to weigh every group element by $\Omega_0(g)$ as we please, but then integrate over all possible transformations between classical reference frame states, which includes integrating the square of (2.31) over $g' \in G$ to reach every initial classical reference frame state, and integrating over $g \in G$ to obtain every transformation possible starting from that initial state. Only in this second integral over $g \in G$ do we introduce Ω_0.

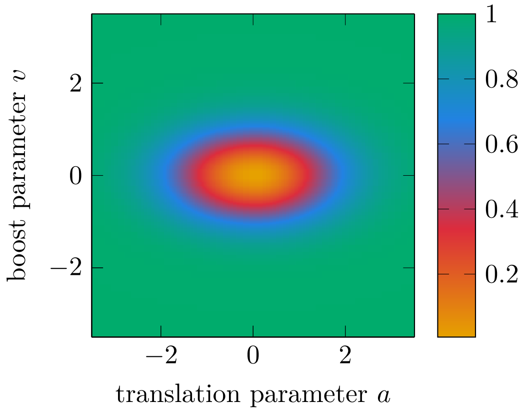


Figure 2.2: A possible weight for a badness measure used to distinguish imperfect reference frames for the Galilei group in one dimension. Here, we give little weight to small translations and boosts (*i.e.* transformations close to e). This makes sense, since small transformations are usually not as crucial to detect compared to large ones. Furthermore, this weight prioritizes accuracy for boosts over translations. Asymmetric weights like this make sense already because position and velocity are not measured in the same units.

Thus, we define:

Definition 2.24: Weighted Badness Measure

A *weight* $\Omega_0 : G \to \mathbb{R}_{\geq 0}$ defines a *weighted badness measure* for imperfect quantum reference frames characterized by their classical states C:

$$B(C, \Omega_0) := \frac{1}{|\langle \Gamma_e | \Gamma_e \rangle|^2} \int_G \mathrm{d}g \, \mathrm{d}g' \, \Omega_0(g) \, \big| \langle \Gamma_e | \, \hat{U}(g'^{-1} g g') \, | \Gamma_e \rangle \, \big|^2. \tag{2.35}$$

We allow formally infinite or infinitesimal weights and badness measures, as long as they can be made finite by multiplying with a formal constant.

This is well-defined: choosing different representatives of C changes the integral by a factor, which is counteracted by the normalization in front of the integral; changing the labelling within a representative as $|\Gamma_e\rangle \rightsquigarrow |\Gamma_{g''}\rangle$ for some $g'' \in G$ amounts to a right-translation in g' which leaves the Haar measure invariant, and thus does not change the integral. Note that the integral over $\mathrm{d}g$ in contrast is *not* invariant under left- or right-translations for general choices of weight Ω_0.

It is possible [19] to obtain an expression for $B(C, \Omega_0)$ similar to our first, flawed idea (2.34):

Proposition 2.25

The badness measure $B(C, \Omega_0)$ can be written as

$$B(C, \Omega_0) = \frac{1}{|\langle \Gamma_e | \Gamma_e \rangle|^2} \int_G \mathrm{d}g \, \Omega(g) \, \big| \langle \Gamma_e | \, \hat{U}(g) \, | \Gamma_e \rangle \, \big|^2, \tag{2.36}$$

where

$$\Omega(g) := \int \mathrm{d}g' \, \Omega_0(g' g g'^{-1}). \tag{2.37}$$

Ω is *conjugation-invariant*:

$$\Omega(g'^{-1}gg') = \Omega(g), \qquad g, g' \in G. \tag{2.38}$$

Thus, Ω is not a function on G, but on the conjugacy classes $\mathrm{conj}(G)$ of G; we also say that Ω is a *class function*.

Conversely, given a class function $\Omega : \mathrm{conj}(G) \to \mathbb{R}_{\geq 0}$,

$$B_{\mathrm{conj}}(C, \Omega) := \frac{1}{|\langle \Gamma_e | \Gamma_e \rangle|^2} \int_G \mathrm{d}g\, \Omega(g) \left| \langle \Gamma_e | \hat{U}(g) | \Gamma_e \rangle \right|^2. \tag{2.39}$$

is a well-defined weighted badness measure.

We show this in appendix A.1. Much of this result, especially the weight being a class function, was already remarked by [19].

So after all, we obtain an expression of the form (2.34) with a weight which cannot always be freely chosen everywhere on G. But, it is *false* that we do not have complete freedom over the weight Ω_0; rather, additional assumptions about our badness measure lead us to naturally consider a certain average Ω of the weight Ω_0, which happens to be a class function. According to preference we can work with the definition (2.35) and a general weight Ω_0, or we can equivalently define the badness measure through (2.39), as characterized by the averaged weight $\Omega(g)$ satisfying (2.38).

For non-compact Lie groups G we have $|G| = \infty$ (see theorem B.4), and every finite Ω_0 will give rise to an infinite Ω, since

$$\Omega(e) = \int_G \mathrm{d}g\, \Omega_0(geg^{-1}) = \int_G \mathrm{d}g\, \Omega_0(e) = |G| \cdot \Omega_0(e). \tag{2.40}$$

This is the reason why we allowed for formally infinite or infinitesimal weights and badness measures in definition 2.24. The infinities involved here are similar to the infinities encountered earlier: both originate in non-compactness of G.

3. Transformations Between Perfect Quantum Reference Frames

As mentioned above, a reference frame should provide a point of view to be taken, and one should be able to transform between these views. In this chapter we construct a framework of transformations between *perfect* quantum reference frames. As we will see, this essentailly results in the framework introduced in [15]; our derivation however differs from the original in two regards: Firstly, we take an "observer-first" approach by starting with the point of view of an observer, a very familiar concept, while they start in the so-called *external view* (which we will discover in section 3.3). Secondly, they work with algebras of observables as well as transformations of observables, in a "Heisenberg-like" picture, we transform states and employ a "Schrödinger-like" picture.

In section 3.1 we begin our "observer-first" approach by defining what states and Hilbert spaces observers have access to, and derive the general form of a quantum reference frame transformation. Section 3.2 then tackles the question of what an observer sees of their own reference frame; it will make sense to assume that they only have access to features which are invariant under G. This assumption then results in the existence of the *external view* in section 3.3, and makes contact with the framework of [15]. We apply the framework in section 3.4 and analyse the structure of states resulting from transformations. Finally, we briefly explore in section 3.5 how the discussed framework contains a simple *perspective-neutral* approach as a special case.

3.1 Observers and Transformations

Our discussion of quantum reference frame transformations begins by describing the Hilbert spaces of any physical system of interest, including reference frames. But whenever we write down the Hilbert space of a quantum system, we also obtain an (experimental) context in which the system can be observed thanks to the observables defined on said Hilbert space. Essentially, writing down a Hilbert space always comes with some implicit choice of "point of view" from which the system is seen. Typically, this point of view is the "laboratory" and we think of ourselves being able to perform the measurements described by observables. But since we wish to develop a theory of such "points of view" using the quantum reference frames introduced in the previous chapter, we must critically reevaluate this position.

Observers. Let us begin with two observers, Alice and Bob. Each observer carries a reference frame, which we call A and B respectively. We will also consider another physical system S which is not necessarily a reference frame, in order to study it from the points of view of both observers. See figure 3.1. Instead of writing down the Hilbert space of the total system from some external laboratory perspective, we directly take the view of either Alice or Bob, and then define the operation of transforming between their views.

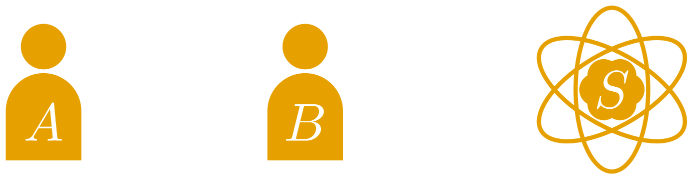


Figure 3.1: Alice A, Bob B and the physical system S they observe.

Let us for instance take the perspective of Alice. We would like Alice to be able to interact with the world as we are used to from standard quantum mechanics. She should thus have full access to the Hilbert space $\mathcal{H}_B$ of Bob's perfect reference frame, and the Hilbert space $\mathcal{H}_S$ of the physical system of interest. But does Alice also have access to a Hilbert space $\mathcal{H}_A$ describing *her own* perfect reference frame? And if she does, what state should her own frame be in? We will not settle these questions here; instead we may assume that Alice has access to $\mathcal{H}_A$, with physically meaningful restrictions on which states are allowed imposed later on, in section 3.2. The same is true for Bob's point of view, except that now the state on $\mathcal{H}_B$ could perhaps be subject to restrictions.

Recall that our group G then acts on $\mathcal{H}_A$ and $\mathcal{H}_B$ through the left-regular representations $\hat{L}_A$ and $\hat{L}_B$ respectively. We assume that G acts on $\mathcal{H}_S$ through a unitary representation $\hat{U}_S$.

For now, we thus have:

Definition 3.1: Observers

Consider two observers Alice (A) and Bob (B), embodying the perfect quantum reference frames $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and let a physical system S of interest be described by the Hilbert space $\mathcal{H}_S$. The group G acts through the unitary representations $\hat{L}_A$, $\hat{L}_B$ and $\hat{U}_S$ on $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_S$ respectively.

Both observers describe physics using states on the Hilbert space

$$\mathcal{H}_{ABS} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_S. \tag{3.1}$$

That is, the state of the total system from Alice's perspective is a density operator

$$\hat{\rho}_{ABS|A} : \mathcal{H}_{ABS} \to \bar{\mathcal{H}}_{ABS}. \tag{3.2}$$

The notation "$|A$" indicates that this state is to be understood from Alice's point of view; the state as seen from Bob is denoted by $\hat{\rho}_{ABS|B}$ and generally different. The state on S is obtained as usual through *partial trace* over all reference frames:

$$\hat{\rho}_{S|A} := \mathrm{tr}_{AB}\big(\hat{\rho}_{ABS|A}\big), \tag{3.3}$$

and analogously for Bob's system state $\hat{\rho}_{S|B}$. Analogously one obtains other reduced states.

The above discussion easily extends to arbitrarily many observers: for each observer we introduce another perfect reference frame factor in the total Hilbert space.

Reference Frame Transformations. Next, we need a *reference frame transformation* taking us from Alice's point of view to that of Bob. We denote this operation by $\mathsf{U}^\dagger_{A\to B}$. Given $\hat{\rho}_{ABS|A}$, it should hold that

$$\hat{\rho}_{ABS|B} = \mathsf{U}^\dagger_{A\to B}[\hat{\rho}_{ABS|A}]. \tag{3.4}$$

Of course, there should also exist the opposite transformation $\mathsf{U}^\dagger_{B\to A}$ such that

$$\mathsf{U}^\dagger_{B\to A} = (\mathsf{U}^\dagger_{A\to B})^{-1}. \tag{3.5}$$

In order to preserve the properties of density operators, we require reference frame transformations to be CPTP [45]; but since they are invertible, they must be *unitary superoperators*:[1]

$$\mathsf{U}^\dagger_{A\to B}[\,\cdot\,] = \hat{U}^\dagger_{A\to B}[\,\cdot\,]\hat{U}_{A\to B}, \tag{3.6}$$

where $\hat{U}^\dagger_{A\to B}$ is unitary.

The "†" in $\mathsf{U}^\dagger_{A\to B}$ and $\hat{U}^\dagger_{A\to B}$ in the notation describes what the transformation is intuitively supposed to do: it should apply the unitary transformation $\hat{U}^\dagger_S(g)$ to the S-part of the state $\hat{\rho}_{ABS|A}$, where $g \in G$ is the transformation required to reach Bob's frame from Alice's frame, and should thus be selected according to the B-part of the state; similar requirements for reference frame transformations are posed in [8]. More precisely, we require that

$$\hat{U}^\dagger_{A\to B} \left|\varphi\right\rangle_A \left|g\right\rangle_B \left|\psi\right\rangle_S = \left|\varphi'\right\rangle_{AB} \hat{U}^\dagger_S(g) \left|\psi\right\rangle_S, \tag{3.7}$$

where $\left|\psi\right\rangle_S$ is any system state, while $\left|\varphi\right\rangle_A$ and $\left|\varphi'\right\rangle_{AB}$ are some currently not further specified states. Figure 3.2 illustrates this assumption for the case $G = (\mathbb{R}, +)$. We will assume that $\left|\varphi'\right\rangle_{AB}$ does not depend on $\left|\psi\right\rangle_S$: the system state should have no effect on the transformation of reference frame states.

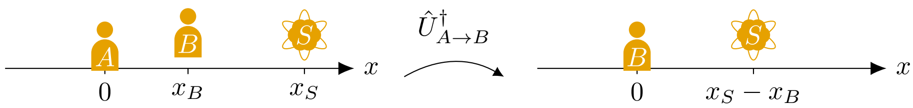


Figure 3.2: Condition (3.7) in the case $G = (\mathbb{R}, +)$ of translations in one dimension, where we can take position eigenstates of a quantum particle as classical reference frame states. Alice sees Bob in a position eigenstate at position x_B. For illustration purposes we also assume that the system is a quantum particle, and that Alice observes it in a position eigenstate at position x_S. The condition (3.7) now ensures that the relative distance between Bob and the system does not change when transforming into Bob's view. Note that we omitted Alice in Bob's view, since we make no assumption about how Bob sees her at this point.

We will further assume that the expression for $\hat{U}^\dagger_{A\to B}$ depends on the type of system S only through the representation $\hat{U}_S$: if we replace the system S with S', then the only change in the expression for $\hat{U}^\dagger_{A\to B}$ is that $\hat{U}_{S'}(g)$ is substituted for $\hat{U}_S(g)$.

Finally, the inverse transformation should be of the same form as $\hat{U}^\dagger_{A\to B}$, except that the roles of A and B are interchanged:

$$\hat{U}^\dagger_{B\to A} = \hat{U}_{A\to B} = \hat{T}_{AB} \cdot \hat{U}^\dagger_{A\to B} \cdot \hat{T}_{AB}, \tag{3.8}$$

where $\hat{T}_{AB}$ is the unitary operator which swaps A and B:

$$\hat{T}_{AB} \left|\psi\right\rangle_A \left|\varphi\right\rangle_B = \left|\varphi\right\rangle_A \left|\psi\right\rangle_B. \tag{3.9}$$

[1] It came to our attention during the final stages of the thesis, that this unfortunately requires some foresight to justify: We did not yet specify the restrictions on the state of an observer's own frame, and so we have not ruled out the possibility of this restriction being in the form of a restriction on the Hilbert space, as is the case in the perspective-neutral approach. This would mean that $\mathsf{U}^\dagger_{A\to B}$ would only have to be unitary on that restricted subspace. We will argue in section 3.2, that it is instead physically reasonable to restrict the set of density operators in a way which does not amount to restricting to a subspace. Thus, unitarity of the full map $\mathsf{U}^\dagger_{A\to B}$ makes sense again.

This requirement can be seen as an incarnation of the *principle of relativity*: every reference frame is to be treated equally [1, 2]. In particular, it must hold that

$$\hat{U}^\dagger_{B\to A} \left|g\right\rangle_A \left|\varphi\right\rangle_B \left|\psi\right\rangle_S = \left|\varphi'\right\rangle_{BA} \hat{U}^\dagger_S(g) \left|\psi\right\rangle_S , \tag{3.10}$$

where $\left|\varphi'\right\rangle_{BA} = \hat{T}_{AB} \left|\varphi'\right\rangle_{AB}$, and thus does not depend on $\left|\psi\right\rangle_S$, and the form of $\hat{U}^\dagger_{B\to A}$ must be independent of the choice of S, up to exchanging the representation $\hat{U}_S$.

These assumptions allow us to further characterize reference frame transformations:

Theorem 3.2: Quantum Reference Frame Transformations

An invertible reference frame transformation $\hat{U}^\dagger_{A\to B} : \mathcal{H}_{ABS} \to \mathcal{H}_{ABS}$ satisfying

(a) $\hat{U}^\dagger_{A\to B} \left|\varphi\right\rangle_A \left|g\right\rangle_B \left|\psi\right\rangle_S = \left|\varphi'\right\rangle_{AB} \hat{U}^\dagger_S(g) \left|\psi\right\rangle_S$, for any $\left|\varphi\right\rangle_A \in \mathcal{H}_A$, $\left|\psi\right\rangle_S \in \mathcal{H}_S$, $g \in G$, and $\left|\varphi'\right\rangle_{AB}$ is a state which depends only on g and $\left|\varphi\right\rangle_A$,

(b) The expression for $\hat{U}^\dagger_{A\to B}$ depends on S only through substituting the corresponding representation $\hat{U}_S$.

(c) $\hat{U}^\dagger_{B\to A} = \hat{T}_{AB} \hat{U}^\dagger_{A\to B} \hat{T}_{AB}$,

is of the form

$$\hat{U}^\dagger_{A\to B} = \int_G \mathrm{d}g\, \mathrm{d}g' \, \left|g^{-1}\right\rangle\left\langle g'\right|_A \otimes \hat{W}(g) \left|g'\right\rangle\left\langle g\right|_B \otimes \hat{U}^\dagger_S(g), \tag{3.11}$$

where $\hat{W}(g)$ is a family of unitary operators parametrized continuously by $g \in G$, and satisfying

$$\hat{W}^\dagger(g) = \hat{W}(g^{-1}), \qquad \forall\, g \in G. \tag{3.12}$$

$\hat{U}^\dagger_{B\to A}$ is given by an analogous expression, with the roles of A and B interchanged.

We prove the theorem in appendix A.2.

Note that the form (3.11) implies

$$\hat{U}^\dagger_{A\to B} \left|\varphi\right\rangle_A \left|g\right\rangle_B \left|\psi\right\rangle_S = \left|g^{-1}\right\rangle_A \left|\varphi''\right\rangle_B \hat{U}^\dagger_S(g) \left|\psi\right\rangle_S , \tag{3.13}$$

where $\left|\varphi''\right\rangle$ is a not further specified state. This fact conveys a kind of *reciprocity* of reference frames: Alice's reference frame appears in the opposite way to Bob, as Bob's frame does to Alice. Figure 3.3 shows this in the context of the example $G = (\mathbb{R}, +)$ of figure 3.2.

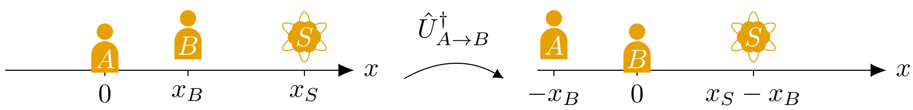


Figure 3.3: The reciprocity of reference frame transformations implied by theorem 3.2 in the case of $G = (\mathbb{R}, +)$ and assuming position eigenstates in Alice's view makes sure that the relative distance between Alice and Bob is kept when transforming into Bob's view. Consequently, Bob sees Alice at the same distance from himself as Alice sees Bob from herself, but on the opposite side.

Theorem 3.2 does not fix the precise form of reference frame transformations: any choice of operators $\hat{W}(g)$ satisfying (3.12) in the theorem, gives rise to a family of reference frame transformations (if we have two observers, there are two transformations, $A \to B$ and $B \to A$) consistent with our assumptions above. The ambiguity present in the choice of

$\hat{W}(g)$ however only impacts the states of reference frames *as seen by their own observer*, *i.e.* the state of A from Alice's point of view and the state of B from Bob's point of view. To further specify $\hat{W}(g)$, we must discuss how observers see the state of their own reference frame. We will do this in the next section.

We will again deal with arbitrarily many observers in a simple manner: any observer which is not part of the reference frame transformation currently considered will simply be treated as part of S. Furthermore, general reference frame transformations can always be built from $\hat{U}^\dagger_{A\to B}$ and various swap operators: for instance, with four observers A, B, C and D we have

$$\hat{U}^\dagger_{D\to C} = \hat{T}_{AD}\hat{T}_{BC}\hat{U}^\dagger_{A\to B}\hat{T}_{BC}\hat{T}_{AD}. \tag{3.14}$$

Finally, we note that a quantum reference frame transformation of the form (3.11) can be rewritten as

$$\hat{U}^\dagger_{A\to B} = \hat{T}_{AB}\int_G \mathrm{d}g\, \hat{W}_A(g) \otimes \left|g^{-1}\right\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g), \tag{3.15}$$

where $\hat{W}_A(g)$ is $\hat{W}$ acting on A. This follows using the completeness relation (2.24).

Examples. Let us consider two examples of families of reference frame transformations:

Example 3.3: Transformation with Complete Reciprocity

One possibility is to assume that observers can measure their own reference frame perfectly. It is then natural to require Alice to see her own reference frame in the opposite way as Bob sees his, *i.e.* to extend the above-mentioned reciprocity to the observer's own frames. This assumption leads us to the following transformation:

$$\hat{U}^\dagger_{A\to B} = \int_G \mathrm{d}g\,\mathrm{d}g'\ \left|g^{-1}\right\rangle\langle g'|_A \otimes \left|g'^{-1}\right\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g). \tag{3.16}$$

Essentially, it swaps and G-inverts the states of A and B while applying the correct transformations to S.

Here, $\hat{W}(g)$ is the unitary *G-inversion* $|g'\rangle \mapsto \left|g'^{-1}\right\rangle$ and thus g-independent. Since the G-inversion is its own inverse, (3.12) is satisfied.

If we do not require observers to measure their own state perfectly, other options are possible. We will follow this approach in the next section. More precisely, we will argue that an observer cannot use their own reference frame to observe their own state, and thus the state must be one which makes sense without a reference frame. We will provide more details in the next section. We however mention already here the transformation introduced by [15], which will also result from our discussion:

Example 3.4: Transformation of [15]

Another possible version of reference frame transformation is

$$\hat{U}^\dagger_{A\to B} = \int_G \mathrm{d}g\,\mathrm{d}g'\ \left|g^{-1}\right\rangle\langle g'|_A \otimes |g'g\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g). \tag{3.17}$$

Now, $\hat{W}(g) = \hat{R}(g^{-1}) = \hat{R}^\dagger(g)$, which satisfies (3.12).

This form of transformation will be useful because it factorizes as

$$\hat{U}^\dagger_{A\to B} = \overbrace{\int_G \mathrm{d}g\, \hat{L}^\dagger_A(g) \otimes |g\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g)}^{=:\, \hat{U}^\dagger_{\to B}} \times$$
$$\times \Bigg[\underbrace{\int_G \mathrm{d}g'\; |g'\rangle\langle g'|_A \otimes \hat{L}^\dagger_B(g') \otimes \hat{U}^\dagger_S(g')}_{=:\, \hat{U}^\dagger_{\to A}}\Bigg]^\dagger, \quad (3.18)$$

which is easily checked by a short computation. We will see in proposition 3.12 that this factorization is a special case of an "almost-factorization" of $\hat{U}^\dagger_{A\to B}$ possible for every reference frame transformation. This special case will give rise to an *external view*, a way of describing physics independently of any observer.

Note also that when written in the form (3.15), this transformation takes the form

$$\hat{U}^\dagger_{A\to B} = \hat{T}_{AB} \int_G \mathrm{d}g\, \hat{R}^\dagger_A(g) \otimes |g^{-1}\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g). \quad (3.19)$$

A state $|\psi\rangle_{AB}$ on AB is a wave function $(g, g') \mapsto \psi_{AB}(g, g')$ in $L^2(G \times G)$; we can similarly view a state $|\chi\rangle_S$ on S as a wave function $x \mapsto \chi_S(x)$ over some (possibly discrete space) X. Denoting the action $\hat{U}_S(g)\,|\chi\rangle_S =: [x \mapsto (\hat{U}_S(g)\chi_S)(x)]$, we find by conditioning (3.19) with $\langle g|_A \langle g'|_B$ that the action of $\hat{U}^\dagger_{A\to B}$ on wave functions is

$$\Big[(g, g', x) \mapsto \psi_{AB}(g, g')\chi_S(x)\Big]$$
$$\overset{A\to B}{\longmapsto} \Big[(g, g', x) \mapsto \psi_{AB}(g' g^{-1}, g^{-1})(\hat{U}_S(g^{-1})\chi_S)(x)\Big] \quad (3.20)$$

Using right-invariance and inversion invariance of the Haar measure, we can directly show using the scalar product (2.15) that this is unitary. This reference frame transformation can thus also be defined completely without referencing delta distributions on G. Similarly, the action of $\hat{U}^\dagger_{\to A}$ is

$$\Big[(g, g', x) \mapsto \psi_{AB}(g, g')\chi_S(x)\Big]$$
$$\overset{\to A}{\longmapsto} \Big[(g, g', x) \mapsto \psi_{AB}(g, g g')(\hat{U}_S(g^{-1})\chi_S)(x)\Big] \quad (3.21)$$

3.2 *G*-Invariance of Reference Frame States

We now turn to the question of what Alice can observe of her own reference frame. We will follow the principle that *Alice cannot use her own reference frame to observe herself.* One option is that she always sees herself in the same way, similarly to how a point particle in classical physics always has a position of zero relative to itself. We briefly investigate this approach in example 3.5 below. This however leaves out the possibility of there being internal information accessible to Alice without the use of a reference frame. We will thus follow a more general approach where we only rule out the degrees of freedom in her state of herself which require a reference frame.

Intermezzo: A More Restrictive Approach. As claimed above, one can allow only a single, special state for Alice's reference frame from her point of view. By relativity, Bob must then also always see that state of himself. This requires some tuning of the general form of reference frame transformations. It can for instance be achieved with the transformations in example 3.3 and the special state $|e\rangle$:

Example 3.5

Assuming that observers see their own reference frame always in the special state $|e\rangle$ is equivalent to restricting their own Hilbert space to $\mathbb{C}$, or even removing it completely, since a single fixed state carries no information. In that scenario, the reference frame transformation from Alice to Bob would be the unitary map $\hat{U}^{\dagger}_{A\to B}$: $\mathcal{H}_A \otimes \mathcal{H}_S \to \mathcal{H}_B \otimes \mathcal{H}_S$ given by

$$\hat{U}^{\dagger}_{A\to B} := \int_G \mathrm{d}g \, \left|g^{-1}\right\rangle_B \langle g|_A \otimes \hat{U}^{\dagger}_S(g). \tag{3.22}$$

I.e., Bob's reference frame gets inverted and becomes Alice's reference frame, while unitaries act accordingly on S.

The transformations in [8] have this form.

***G*-Invariance and the *G*-Twirl.** Let us first consider the more general problem determining which quantum states are physical in the absence of a reference frame. Let $\mathcal{H}$ be a Hilbert space carrying a representation $\hat{U}$ of G. It is generally accepted [15–22] that *in the absence of a reference frame for G, $\hat{\rho}$ must be G-invariant*, that is,

$$\hat{U}(g)\,\hat{\rho}\,\hat{U}^{\dagger}(g) = \hat{\rho}, \qquad \forall\, g \in G, \tag{3.23}$$

which is equivalent to

$$\big[\hat{U}(g), \hat{\rho}\big] = 0, \qquad \forall\, g \in G. \tag{3.24}$$

This is sometimes called the *maximum entropy principle* [19], since G-invariant states are completely mixed with respect to the action of G in a sense that we will explain soon. States which are not G-invariant contain *unspeakable information* [19] which appears as absolute information in the absence of a reference frame, breaking the principle of relativity.

Let us see this idea at work with a simple example:

Example 3.6

Take $G = (\mathbb{R}, +)$ and consider $\mathcal{H}_S = L^2(\mathbb{R})$. In the absence of a reference frame for translations, it makes no sense to *e.g.* say that S is in the position eigenstate $\rho_S = |x\rangle\langle x|_S$. Essentially, we cannot speak of position if we do not know where the origin is. And nevertheless speaking of position would imply an *absolute position*, in violation of the principle of relativity.

More generally, any piece of information extractable from the state of S which is not invariant under translations could be used to deduce some amount of absolute position information. Hence, in the absence of a reference frame for G, we must assume that the state $\hat{\rho}_S$ on S is G-invariant.

With a reference frame present, it again makes sense to talk about state ρ_S which are not G-invariant, since they are understood relative to the reference frame.

Also, G-invariance generalizes to multipartite states: if $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with the representation $\hat{U}_1$ of G acting on $\mathcal{H}_1$, we say that a state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is *G-invariant on $\mathcal{H}_1$* if

$$\left[\hat{U}_1(g) \otimes \hat{\mathrm{id}}_2, \hat{\rho}\right] = 0, \qquad \forall\, g \in G. \tag{3.25}$$

A very useful tool for the manipulation of G-invariant states is the *G-twirl* [19]:

Definition 3.7: G-twirl

Let $\mathcal{H}$ be a Hilbert space with unitary representation $\hat{U}$ of G. The *G-twirl* is the superoperator

$$\mathsf{G}[\,\cdot\,] := \frac{1}{|G|} \int_G \mathrm{d}g\, \hat{U}(g)\, [\,\cdot\,]\, \hat{U}^\dagger(g). \tag{3.26}$$

For any state $\hat{\rho}$ on $\mathcal{H}$ its G-twirl is $\mathsf{G}[\hat{\rho}]$.

On the bipartite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with representation $\hat{U}_1$ acting on $\mathcal{H}_1$, we define the *G-twirl on $\mathcal{H}_1$* as

$$\mathsf{G}_1 := \mathsf{G} \otimes \mathsf{id}_2, \tag{3.27}$$

where G is the G-twirl on $\mathcal{H}_1$ defined above (using $\hat{U}_1$ in place of $\hat{U}$).

Typically, we will have $\mathcal{H}_1 = \mathcal{H}_A$ with $\hat{U} = \hat{L}_A$, *etc.* We then also speak of the "G-twirl on A" or the "G-twirl on Alice's frame", *etc.* The G-twirl is useful thanks to its properties:

Proposition 3.8: Properties of the G-Twirl

Let $\mathcal{H}$ be a Hilbert space with unitary representation $\hat{U}$.

(a) G is CPTP.

(b) The G-twirl is a projector: $\mathsf{G}^2 = \mathsf{G}$.

(c) The G-twirl $\mathsf{G}[\hat{\rho}]$ of any state $\hat{\rho}$ is G-invariant.

(d) $\mathsf{G}[\hat{\rho}] = \hat{\rho}$ if and only if $\hat{\rho}$ is G-invariant.

These properties also hold for G_1 acting on the first factor of the bipartite Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ if one replaces "G-invariant" with "G-invariant on $\mathcal{H}_1$". A direct consequence of (a) is that for all states $\hat{\rho}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\mathsf{tr}_1\big(\mathsf{G}_1[\hat{\rho}]\big) = \mathsf{tr}_1(\hat{\rho}). \tag{3.28}$$

A proof is provided in appendix A.3.

Essentially, G projects onto the subset of states which are G-invariant. This makes the G-twirl an important tool when dealing with reference frame states as seen from the reference frame itself, since these must be G-invariant. Instead of working with G-invariant states we can now also work with G-twirled general states.

Refining the Reference Frame Transformations. As stated above, we allow only those states of Alice in her own perspective, which can exist without a reference frame. With the G-twirl this requirement is simply

$$\mathsf{G}_A\left[\hat{\rho}_{ABS|A}\right] = \hat{\rho}_{ABS|A}, \tag{3.29}$$

where the G-twirl uses the left-regular representation $\hat{L}_A$. The same must of course hold for Bob's state $\hat{\rho}_{ABS|B}$, but with G_A replaced by G_B. Any reference frame transformation

$\mathsf{U}^\dagger_{A\to B}$ must thus conserve this property, *i.e.*

$$\mathsf{U}^\dagger_{A\to B} \circ \mathsf{G}_A = \mathsf{G}_B \circ \mathsf{U}^\dagger_{A\to B}. \tag{3.30}$$

Besides the requirements in theorem 3.2 we take this as an additional requirement for quantum reference frame transformations. We can then show that:

Proposition 3.9: Conservation of G-Invariance

Equation (3.30) holds if and only if for all $g, g' \in G$

$$\hat{W}(g)\,\mathsf{G}[\,\cdot\,]\,\hat{W}^\dagger(g') = \mathsf{G}\big[\hat{W}(g)\,[\,\cdot\,]\,\hat{W}^\dagger(g')\big]. \tag{3.31}$$

The proof is provided in appendix A.4.

Let us revisit the examples 3.3 and 3.4 of reference frame transformations and see whether they satisfy proposition 3.9:

Example 3.10: Example 3.3 Generally Does not Conserve G-Invariance

In example 3.3, $\hat{W}(g) = \hat{W}$ does not depend on g and is given by the inversion map $g' \mapsto g'^{-1}$, in particular satisfying $\hat{W}^\dagger = \hat{W}$. For some $g \in G$ we now insert $\hat{R}(g)$ into (3.31). Using proposition 2.14 and thus $\mathsf{G}[\hat{R}(g)] = \hat{R}(g)$, the left-hand side is then

$$\hat{W}\hat{R}(g)\hat{W} = \hat{L}(g), \tag{3.32}$$

which can easily be checked by computing the action on $|g'\rangle$. The right-hand side is

$$\mathsf{G}[\hat{L}(g)] = \frac{1}{|G|}\int_G \mathrm{d}g'\,\hat{L}(g'gg'^{-1}). \tag{3.33}$$

If G is Abelian, then the left- and right-hand sides are equal for all $g \in G$, because $g'gg'^{-1} = g'g'^{-1}g = g$. If G is not Abelian, then there exist $g, g' \in G$ such that $g'gg'^{-1} \neq g$, and hence the two sides are not equal. Thus, the family of reference frame transformations in example 3.3 conserve G-invariance if and only if G is Abelian.

Example 3.11: Example 3.4 Conserves G-Invariance

Clearly, a sufficient condition for (3.31) is

$$\big[\hat{W}(g), \hat{L}(g')\big] = 0, \qquad \forall\, g, g' \in G. \tag{3.34}$$

In example 3.4, we have $\hat{W}(g) = \hat{R}^\dagger(g) = \hat{R}(g^{-1})$, and thanks to proposition 2.14, (3.34) holds. Consequently, the family of reference frame transformations in example 3.4 conserves G-invariance.

The failure of example 3.3 in the case of a non-Abelian group is closely linked to the existence of states which are invariant under $\hat{L}$, but not under $\hat{R}$ (so-called L-invariant but not R-invariant states), and *vice-versa*. We investigate this connection in appendix B.4.

3.3 External View

While we have excluded certain reference frame transformations such as those in example 3.3 by requiring conservation of G-invariance, there are still many remaining choices, example

3.4 among them. We will now see another criterion by which one may distinguish various reference frame transformations: the existence of an *external view*.

Factorization of Reference Frame Transformations. We begin by noting that all reference frame transformations (including those which do not preserve G-invariance) can be "almost"-factorized into two parts, and truly factorized in the case of example 3.4:

Proposition 3.12: (Almost-) Factorization of Frame Transformations

The reference frame transformation $\hat{U}^\dagger_{A\to B}$ can be written as

$$\hat{U}^\dagger_{A\to B} = \int_G \mathrm{d}g\, \mathrm{d}g'\, \hat{L}^\dagger_A(g)\, |g'\rangle\langle g'|_A \otimes \hat{W}(g'^{-1}g)\, |g'\rangle\langle g|_B\, \hat{L}_B(g') \otimes \hat{U}^\dagger_S(g)\hat{U}_S(g') \quad (3.35)$$

If $\hat{W}(g'^{-1}g)\,|g'\rangle = |g\rangle$, *i.e.* if $\hat{W}(g) = \hat{R}^\dagger(g)$, then $\hat{U}^\dagger_{A\to B}$ factorizes as

$$\hat{U}^\dagger_{A\to B} = \hat{U}^\dagger_{\to B} \cdot \hat{U}_{\to A}, \quad (3.36)$$

where

$$\hat{U}^\dagger_{\to A} := \int_G \mathrm{d}g\; |g\rangle\langle g|_A \otimes \hat{L}^\dagger_B(g) \otimes \hat{U}^\dagger_S(g) \quad (3.37)$$

$$\hat{U}^\dagger_{\to B} := \int_G \mathrm{d}g\; \hat{L}^\dagger_A(g) \otimes |g\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g) = \hat{T}_{AB} \cdot \hat{U}^\dagger_{\to A} \cdot \hat{T}_{AB}. \quad (3.38)$$

Proof. The expression for $\hat{U}^\dagger_{A\to B}$ from theorem 3.2 can be rewritten as

$$\begin{aligned}\hat{U}^\dagger_{A\to B} &= \int_G \mathrm{d}g\,\mathrm{d}g'\; \left|g^{-1}\right\rangle\langle g'|_A \otimes \hat{W}(g)\,|g'\rangle\langle g|_B \otimes \hat{U}^\dagger_S(g) \\ &= \int_G \mathrm{d}g\,\mathrm{d}g'\; \left|g^{-1}g'\right\rangle\langle g'|_A \otimes \hat{W}(g'^{-1}g)\,|g'\rangle\left\langle g'^{-1}g\right|_B \otimes \hat{U}^\dagger_S(g'^{-1}g) \\ &= \int_G \mathrm{d}g\,\mathrm{d}g'\; \hat{L}^\dagger_A(g)\,|g'\rangle\langle g'|_A \otimes W(g'^{-1}g)\,|g'\rangle\langle g|_B\, \hat{L}_B(g') \otimes \hat{U}^\dagger_S(g)\hat{U}_S(g'), \end{aligned} \quad (3.39)$$

showing the general case in the proposition. The special case is then easily checked. □

For later it will be useful to introduce the superoperators

$$\mathsf{U}^\dagger_{\to A}[\,\cdot\,] := \hat{U}^\dagger_{\to A}[\,\cdot\,]\,\hat{U}_{\to A}, \quad (3.40)$$

$$\mathsf{U}_{\to A}[\,\cdot\,] := \hat{U}_{\to A}[\,\cdot\,]\,\hat{U}^\dagger_{\to A}, \quad (3.41)$$

and analogously for $\mathsf{U}^\dagger_{\to B}$ and $\mathsf{U}_{\to B}$. With this we can write the reference frame transformations of example 3.3 as

$$\mathsf{U}^\dagger_{A\to B} = \mathsf{U}^\dagger_{\to B} \circ \mathsf{U}_{\to A}, \quad (3.42)$$

and similarly for the inverse transformation. Also, note that $\hat{U}^\dagger_{\to A}$ does not differentiate between B and S. In fact, it will sometimes be useful to consider a more general Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_Q$, where $\mathcal{H}_Q$ is any Hilbert space equipped with a unitary representation $\hat{U}_Q$ of G, and define on that space

$$\hat{U}^\dagger_{\to A} := \int_G \mathrm{d}g\; |g\rangle\langle g|_A \otimes \hat{U}^\dagger_Q(g). \quad (3.43)$$

Similarly, $\hat{U}^\dagger_{\to B}$ does not differentiate between A and S and one can analogously generalize $\hat{U}^\dagger_{\to B}$, or for any other reference frame for that matter. The definitions of $\hat{U}^\dagger_{\to A}$ and $\hat{U}^\dagger_{\to B}$

in proposition 3.12 are then just special cases of this definition, with $\mathcal{H}_Q = \mathcal{H}_B \otimes \mathcal{H}_S$ and $\mathcal{H}_Q = \mathcal{H}_A \otimes \mathcal{H}_S$ respectively.

Importantly, physical states, *i.e.* those which are G-invariant on the reference frame from the point of view of the corresponding observer, become *completely G-invariant* once $\mathsf{U}_{\to A}$ is applied [15]:

Theorem 3.13

It holds that

$$\mathsf{G}_{AQ} \circ \mathsf{U}_{\to A} = \mathsf{U}_{\to A} \circ \mathsf{G}_A, \tag{3.44}$$

and equivalently

$$\mathsf{U}^\dagger_{\to A} \circ \mathsf{G}_{AQ} = \mathsf{G}_A \circ \mathsf{U}^\dagger_{\to A}. \tag{3.45}$$

G_{AQ} is the G-twirl on the whole system, *i.e.* using the representation $\hat{L}_A \otimes \hat{U}_Q$.

In particular, this holds for $Q = BS$; and of course, one may replace A by B or by any other reference frame. We reproduce the proof of [15] in appendix A.5.

External View. $\hat{U}^\dagger_{\to B}$ can be interpreted as selecting G-transformations according to the state of B, and applying them to A and S; similarly $\hat{U}^\dagger_{\to A}$ selects transformations according to A and applies them to B and S. Both can thus be seen as some kind of reference frame transformations of their own; not quite, because they do not have the form required be theorem 3.2. We will come back to this feature later.

Still, we can interpret the factorization $\hat{U}^\dagger_{A\to B} = \hat{U}^\dagger_{\to B} \cdot \hat{U}_{\to A}$ in case of example 3.4 as first "jumping out of A" to an intermediate representation of the quantum state through the application of $\hat{U}_{\to A}$, and then "jumping into B" through the application of $\hat{U}^\dagger_{\to B}$. As for reference frame transformations we include "†" in our notation to show how G acts on S "when jumping in the direction of the arrow $_\to$". The intermediate representation is a shared midpoint for all possible reference frame transformations of any given state. We call the intermediate representation of quantum states the *external view.*[2]

Thanks to theorem 3.13 physical states are completely G-invariant in the external view. As explained earlier at the beginning of section 3.2, completely G-invariant states are considered to be the only physical states in cases where no reference frame for G is available. In this sense, the external view can be seen as an *observer-independent* description of the whole physical system.

The specific reference frame transformation of example 3.4 has first been derived in [15], starting with their version of the external view.[3] Our approach so far began with the views of observers and the general form of reference frame transformations between then; the external view was then found as an interesting occurrence in a special case. We will now argue that the occurrence of the external view is crucial. This will allow us to once and for all single out the reference frame transformations of example 3.4, and show that our external view cn also be seen as the external view of [15].

Forgetting about a Reference Frame. Since a state must be completely G-invariant in the absence of a reference frame, we may also attempt to interpret the states in the external view as resulting from physical states which were once understood relative to a third observer, Charlie C, with perfect reference frame $\mathcal{H}_C$, before we "forgot" about Charlie. In

[2] As we will see, [15] also has a notion of external view, although a slightly different one. Moreover, they begin their discussion by assuming this external view. To make full contact with the version of external view in [15], some more work will be required, however ours and their notion will be compatible.

[3] They denote $\hat{U}^\dagger_{\to A}$ by "$\hat{U}^\dagger_{BS}(g_A)$", and $\hat{U}^\dagger_{\to B}$ by "$\hat{U}^\dagger_{AS}(g_B)$".

order to "forget" Charlie we first trace out his reference frame and then perform a G-twirl on the remaining system ABS, because the reference frame we used to observe the state has just been removed. Alternatively, one can also G-twirl $ABCS$ first and trace out C later. As it turns out, these two version of forgetting are equivalent:

Proposition 3.14: Forgetting about a Reference Frame

Consider the Hilbert space $\mathcal{H}_C \otimes \mathcal{H}_Q$, where $\mathcal{H}_C \cong L^2(G)$ is a perfect reference frame for G and $\mathcal{H}_Q$ is any Hilbert space carrying a unitary representation of G.[4] The operation of *forgetting* C, denoted by F_C, can then be achieved by two equivalent operations:

$$\mathsf{F}_C := \mathsf{G}_Q \circ \mathsf{tr}_C = \mathsf{tr}_C \circ \mathsf{G}_{CQ}, \tag{3.46}$$

resulting in a completely G-invariant state on $\mathcal{H}_Q$.

The proof can be found in appendix A.6.

Note that the notion of forgetting about an observer and the resulting completely G-invariant states always occur, irrespective of whether our reference frame transformations factor into two jumps and hence irrespective of whether an external view exists. Now, after having forgotten about an observer, we would like to be able to carry on with doing reference frame transformations using the remaining observers. After all, ignoring one out of potentially arbitrarily many observers should not hinder the functionality of the remaining ones. So for any remaining observer, say Alice A, there must be a map $\mathsf{V}^\dagger_{\to A}$ which turns the completely G-invariant state left over after forgetting Charlie into a state in Alice's perspective. In other words, $\mathsf{V}^\dagger_{\to A}$ must satisfy a requirement analogous to that of $\mathsf{U}^\dagger_{\to A}$ described in theorem 3.13, namely

$$\mathsf{V}^\dagger_{\to A} \circ \mathsf{G}_{AQ} = \mathsf{G}_A \circ \mathsf{V}^\dagger_{\to A}. \tag{3.47}$$

A priori, $\mathsf{V}^\dagger_{\to A}$ must only be CPTP, not necessarily unitary. If it is not unitary however, then there is further information lost when applying $\mathsf{V}^\dagger_{\to A}$. Now this information can only pertain to Charlie, since the functioning of all other frames and/or the remaining physical system cannot be hindered by F_C; but this would mean that F_C has not completely removed all traces of C. It thus makes sense to assume that $\mathsf{V}^\dagger_{\to A}$ is unitary. This then immediately implies that any reference frame transformation factors into two jumps, and there exists an external view as intermediate stage between the jumps:

$$\mathsf{U}^\dagger_{A\to B} = \mathsf{V}^\dagger_{\to B} \circ \mathsf{V}_{\to A}, \qquad \mathsf{V}^\dagger_{\to B} = \mathsf{T}_{AB} \circ \mathsf{V}^\dagger_{\to A} \circ \mathsf{T}_{AB}, \tag{3.48}$$

where $\mathsf{T}_{AB}[\,\cdot\,] = \hat{T}_{AB}[\,\cdot\,]\hat{T}_{AB}$ is the unitary superoperator which swaps the systems A and B. The form of $\mathsf{V}^\dagger_{\to A}$ can be restricted even more: in line with the principle of relativity, $\mathsf{V}^\dagger_{\to A}$ is only allowed to treat the subsystem A in a special way, but must act equally on all systems contained in Q. Thus, we have found a very strong motivation for the existence of the external view. We will assume that the external view exists and that $\mathsf{U}^\dagger_{A\to B}$ factors as in (3.48). Of course, the reference frame transformations in example 3.4 satisfy these demands, with $\mathsf{V}^\dagger_{\to A} = \mathsf{U}^\dagger_{\to A}$. Figure 3.4 summarizes the relationship between forgetting an observer, the external view and the jumps into reference frames.

There are currently two ways of arriving in the external view: one can jump to it from a frame A by means of $\mathsf{V}_{\to A}$, or one can consider a situation with an additional frame C and forget about C through F_C. Let us thus ask how they are related. Recall that F_C can be seen as a G-twirl followed by tracing out C. This implies that whatever information is contained in every other part of the external view state is not related to C. Hence, it makes sense to require that tracing out C in the external view (of the three observers A, B and C)

[4] In the presently discussed situation, we would have $\mathcal{H}_Q = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_S$.

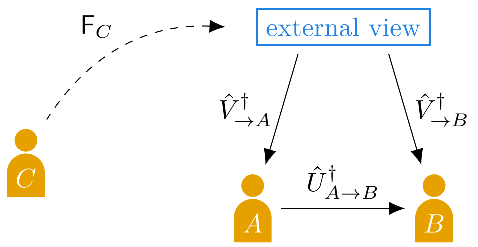


Figure 3.4: Forgetting the observer C through F_C leads us to the external view, from where we can reach Alice's or Bob's view through the unitary jumps $\hat{V}^\dagger_{\to A}$ and $\hat{V}^\dagger_{\to B}$. The reference frame transformation $\hat{U}^\dagger_{A\to B}$ consequently factors as $\hat{U}^\dagger_{A\to B} = \hat{V}^\dagger_{\to B} \cdot \hat{V}_{\to A}$.

has the same effect as forgetting C:

$$\mathsf{F}_C = \mathsf{tr}_C \circ \mathsf{V}_{\to C} \circ \mathsf{G}_C. \tag{3.49}$$

This is the sense in which we take the two ways into the external view to be related. Technically this must only hold for states which are G-invariant on C, hence the G-twirl on C on the left-hand side.[5] As it turns out, the reference frame transformations of example 3.4 also satisfies this second requirement:

Proposition 3.15: Jumps and Forgetting Observers

$\mathsf{V}_{\to C} = \mathsf{U}_{\to C}$ satisfies (3.49).

We provide the proof in appendix A.7.

Uniqueness of Reference Frame Transformations. Having argued for the existence of the external view by considering the act of forgetting about an observer, we now turn to the question of uniqueness. Example 3.4 satisfies all our demands, and in some sense, it is also the only possible reference frame transformation:

Theorem 3.16: Uniqueness of Reference Frame Transformations

Let $\hat{U}^\dagger_{A\to B}$ be a reference frame transformation satisfying the requirements of theorem 3.2, and which gives rise to an external view through the factorization

$$\mathsf{U}^\dagger_{A\to B} = \mathsf{V}^\dagger_{\to B} \circ \mathsf{V}_{\to A}, \qquad \mathsf{V}^\dagger_{\to B} = \mathsf{T}_{AB} \circ \mathsf{V}^\dagger_{\to A} \circ \mathsf{T}_{AB}, \tag{3.50}$$

where $\mathsf{V}^\dagger_{\to A}[\,\cdot\,] = \hat{V}^\dagger_{\to A}[\,\cdot\,]\hat{V}_{\to A}$ is unitary, acts on all subsystems besides A in the same way, and satisfies

$$\mathsf{V}^\dagger_{\to A} \circ \mathsf{G}_{AQ} = \mathsf{G}_A \circ \mathsf{V}^\dagger_{\to A}, \tag{3.51}$$

with Q denoting the subsystems besides A. (Note that this implies (3.30), *i.e.* $\mathsf{U}_{A\to B}$ conserves G-invariance of the observer's own frames).

Then $\hat{U}^\dagger_{A\to B}$ must be of the form

$$\hat{U}^\dagger_{A\to B} = \hat{T}_{AB}\hat{X}\hat{T}_{AB} \cdot \hat{U}^\dagger_{\to B}\hat{U}_{\to A} \cdot \hat{X}, \tag{3.52}$$

where $\hat{X} = \hat{X}_A \otimes \hat{\mathrm{id}}_Q$ is a unitary only acting on A.

[5] In fact, without this G-twirl the requirement would be too strong.

Furthermore, compatibility of jumping into the external view and of forgetting observers follows:

$$\mathsf{F}_A = \mathrm{tr}_A \circ \mathsf{V}_{\to A} \circ \mathsf{G}_A. \tag{3.53}$$

The unitary $\hat{X}_A$ can be interpreted as a change of basis on observer's own reference frames, *agreed upon by every observer*. Thus, $\hat{X}_A$ does not change the physics of reference frame transformations, and we can without loss of generality take $\hat{X}_A = \hat{\mathrm{id}}_A$.

The proof can be found in appendix A.8. With this result, we can without loss of generality stick to the reference frame transformations of example 3.4.

Note that the theorem does not further characterize the possible unitaries $\hat{X}_A$, since we are mostly happy with the choice $\hat{X}_A = \hat{\mathrm{id}}_A$ which is as good as any other. Let us nevertheless make some remarks about possible $\hat{X}_A$. Firstly, a careful examination of the proof in appendix A.8 shows that we must have

$$\hat{W}(g) = \hat{X}_A^\dagger \hat{R}^\dagger(g) \hat{X}_A, \tag{3.54}$$

which must satisfy (3.12); but this is always the case. Secondly, $\hat{U}^\dagger_{A \to B}$ must satisfy (3.30). One can show that this amounts to requiring

$$\mathsf{X}_A \circ \mathsf{G}_A = \mathsf{G}_A \circ \mathsf{X}_A, \tag{3.55}$$

where $\mathsf{X}_A[\,\cdot\,] = \hat{X}_A[\,\cdot\,]\hat{X}_A^\dagger$. This is for instance trivially satisfied (with $\mathsf{X}_A \circ \mathsf{G}_A = \mathsf{G}_A \circ \mathsf{X}_A = \mathsf{G}_A$) if $\hat{X}_A$ is any left-regular action. A non-trivial solution would be if $\hat{X}_A$ is any right-regular action and G is non-Abelian, since then there are states which are (left-) G-invariant, but not right-G-invariant (see appendix B.4).

The External View of [15]. In [15], the external view is understood as the view of an external observer, who needs no reference frame to observe physics. From there, jumps into actual reference frames are derived (notably, resulting in the transformations of example 3.4), and it is argued that the resulting framework is independent of the choice of this external observer. Importantly, the change from the external perspective into the "internal" perspectives of reference frames such as Alice and Bob involves a G-twirl in the external view (or equivalently, a G-twirl on the observer's own reference frame states in their view, see theorem 3.13).

Their external observer can be seen as our Charlie, with the difference that we go a step further and model Charlie as an actual observer described by their very own reference frame. We can make Charlie an "external observer" in the sense of [15], by tracing him out (without G-twirl). The step back into the "internal" view is then made through the G-twirl, and hence we recover our notion of forgetting Charlie.

Finally, the independence of the framework [15] from the external observer is a result comparable to our interpretation of external view states being observer-independent. In summary, the subtle difference between our approach and that of [15] is that for us, the "external view" only holds G-invariant states, while the version of "external view" employed in [15] holds general states seen by an external observer, which first have to be G-twirled prior to jumping into an actual reference frame.

The External View as a Lab Frame. Consider again a state $\hat{\rho}_{ABCS|C}$ in Charlie's view. But instead of forgetting him, we now simply trace him out to get the non-G-invariant state $\hat{\rho}_{ABS} := \mathrm{tr}_C \hat{\rho}_{ABCS|C}$. In [15], this would be a state seen by the external observer Charlie. Does this state have any physical interpretation for us besides this?

To answer this question we note the following:

Proposition 3.17: Alice's Point of View on S

It holds that

$$\mathsf{tr}_{AB} \circ \mathsf{U}^\dagger_{\rightarrow A}[\,\cdot\,] = \mathsf{tr}_{AB} \circ \mathsf{U}^\dagger_{B\rightarrow A}[\,\cdot\,]$$
$$= \int_G \mathrm{d}g \,\big(\langle g|_A \otimes \hat{U}^\dagger_S(g)\big)\,\mathsf{tr}_B[\,\cdot\,]\,\big(|g\rangle_A \otimes \hat{U}_S(g)\big). \tag{3.56}$$

Also, performing G_A just before tr_{AB} in either case does not change the outcome. And the same results hold if we interchange the roles of Alice and Bob (*i.e.* replace $A \leftrightsquigarrow B$).

We prove this in appendix A.9.

Essentially, jumping from external view into Alice's frame or transforming there from another observer's frame is equivalent, when one is only interested in the outcome on the system S. In this sense we can "abuse" the external view as a reference frame to prepare states in, with the intention to then jump to actual reference frames, if we are finally only interested in S; such a state may for instance be prepared by the procedure $\hat{\rho}_{ABS} := \mathsf{tr}_C \hat{\rho}_{ABCS|C}$ explained above. Since we do not follow the rule of G-twirling after the observer is removed, we can think of the external view as some kind of "lab frame" where a reference frame is not necessary.

We want to stress that the interpretation of general states in the external view as a reference frame only works indirectly, through Alice's view on S after a jump from the external view. The interpretation of G-invariant states in the external view as observer-independent states for comparison holds in any circumstance.

Proposition 3.17 of course also clearly illustrate how transformations and jumps work when considering S alone: the group elements $g \in G$ selected by conditioning on A are applied through $\hat{U}^\dagger_S(g)\,[\,\cdot\,]\,\hat{U}_S(g)$ on the system S, as we stressed many times in different contexts. The resulting system state is typically mixed:

Example 3.18

Let $|\phi\rangle_A = |g_1\rangle + |g_2\rangle$, where $g_1, g_2 \in G$, $g_1 \neq g_2$, let σ_B be a state with unit trace, and let $\hat{\varsigma}_S$ be any state. Set $\hat{\rho}_{ABS} = |\phi\rangle\langle\phi|_A \otimes \sigma_B \otimes \hat{\varsigma}_S$. From proposition 3.17 it then follows that irrespective of whether we use $\hat{\rho}_{ABS}$ as initial state for a jump to Alice or a transformation to her, she will observe the system state

$$\hat{\rho}_{S|A} = \delta(e)\left[\hat{U}^\dagger_S(g_1)\hat{\varsigma}_S\hat{U}_S(g_1) + \hat{U}^\dagger_S(g_2)\hat{\varsigma}_S\hat{U}_S(g_2)\right]. \tag{3.57}$$

If the two terms are not proportional then $\hat{\rho}_{S|A}$ is mixed. The factor $\delta(e)$ is due to $\mathsf{tr}\,|g\rangle\langle g| = \delta(e)$.[6]

Such mixtures of system states are a feature exhibited only by quantum reference frames: After the switch (jump or transformation) to Alice's frame, AB holds the information about the state of Alice's frame before the switch, which in the example above was in a superposition of classical states. We will now investigate this feature further, by no longer just considering S.

[6]Recall from section 2.4 that we can deal with such formally infinite factors as we would with any other symbol and so this is not further problematic. See appendix B.3 for details on how we treat such infinities as well as states scaled by them.

3.4 Structure of States & Relativity of Entanglement

Let us now apply the reference frame transformations we derived in the preceding sections to physically interesting situations and investigate the states resulting from transformations. We assume that Alice's state is a product state

$$\hat{\rho}_{ABS|A} := \mathsf{G}_A[\hat{\sigma}_A] \otimes |\varphi\rangle\langle\varphi|_B \otimes \hat{\varsigma}_S, \tag{3.58}$$

where the A- and S-parts are general, but Bob's reference frame state $|\varphi\rangle_B = \int_G \mathrm{d}g\, \varphi(g)\, |g\rangle_B$ is pure. To declutter the notation we will usually not worry about normalization of either the total state or the reduced states, since it can always be achieved with at most formally infinite constants. Jumping into Bob's frame results in

$$\hat{\rho}_{ABS|B} = \mathsf{U}^\dagger_{A\to B}[\hat{\rho}_{ABS|B}] = \int_G \mathrm{d}g_1\, \mathrm{d}g_2\, \langle g_1|\varphi\rangle\langle\varphi|g_2\rangle \times \\ \times \left|g_1^{-1}\right\rangle\!\left\langle g_2^{-1}\right|_A \otimes \hat{R}^\dagger_B(g_1)\mathsf{G}_B[\hat{\sigma}_B]\hat{R}_B(g_2) \otimes \hat{U}^\dagger_S(g_1)\,\hat{\varsigma}_S\,\hat{U}_S(g_2). \tag{3.59}$$

Here, $\hat{\sigma}_B$ is $\hat{\sigma}_A$ taken as a density operator on B instead of A. This expression is most readily derived from the form (3.19) of the transformation. Let us analyse the structure of the state (3.59) more closely.

State on BS. Tracing out A is quite easy, yielding

$$\hat{\rho}_{BS|B} = \int_G \mathrm{d}g \left|\,\langle g|\varphi\rangle\,\right|^2 \cdot \hat{R}^\dagger_B(g)\mathsf{G}_B[\hat{\sigma}_B]\hat{R}_B(g) \otimes \hat{U}^\dagger_S(g)\,\hat{\varsigma}_S\,\hat{U}_S(g). \tag{3.60}$$

Now $g \mapsto |\,\langle g|\varphi\rangle\,|^2$ is up to a positive constant a probability distribution (it is one if $\langle\varphi|\varphi\rangle = 1$ thanks to (2.24)), and $\hat{\rho}_{BS|B}$ is thus a *separable*, hence *non-entangled* state [45].

State on AS & Relativity of Entanglement That there is no entanglement between B and S is not surprising; if at all present, we would expect it between A and S, since B controls the unitaries on S during the transformation, and the state on A after the transformation reflects the state of B before. For transformations of the type of example 3.5, this is indeed the case [15]. This phenomenon is called the *relativity of entanglement*: a state that one observer sees as a product state may be an entangled state in the point of view of another.

To check for entanglement between A and S in our more general framework, we trace out B. Without more assumptions on $\mathsf{G}_\mathsf{B}[\hat{\sigma}_B]$ this is unfortunately not straightforward. But we can already learn much from two extreme cases.

Firstly, take the standpoint that Alice knows nothing about her own state. Hence, we would have $\hat{\sigma}_A = \hat{\mathrm{id}}_A$, which is G-invariant ($\mathsf{G}_A[\hat{\mathrm{id}}_A] = \hat{\mathrm{id}}_A$) and thus a valid state. Consequently, $\hat{R}^\dagger_B(g_1)\mathsf{G}_\mathsf{B}[\hat{\sigma}_B]\hat{R}_B(g_2) = \hat{R}(g_1^{-1}g_2)$. Thus, tracing out B forces $g_1 = g_2$ as before, and[7]

$$\hat{\rho}_{BS|B} = |G| \int_G \mathrm{d}g \left|\,\langle g|\varphi\rangle\,\right|^2 \cdot \left|g^{-1}\right\rangle\!\left\langle g^{-1}\right|_A \otimes \hat{U}^\dagger_S(g)\,\hat{\varsigma}_S\,\hat{U}_S(g). \tag{3.61}$$

As we can see, this is again separable, hence *not entangled*. Even in the case where we do not care at all about Alice's own state we do not find relativity of entanglement as in [8]; the simple presence of a state describing Alice from her point of view seems to prevent it.

Secondly, we would like to find another extreme, where entanglement occurs. Because it is generally very difficult to confirm entanglement in mixed states (see *e.g.* [48]), we will

[7]Here it is perhaps instructive to consider normalization. For proper normalization, we should use $\hat{\mathrm{id}}_A/\mathsf{tr}\,\hat{\mathrm{id}}_A$ instead of $\hat{\mathrm{id}}_A$. Now, $\mathsf{tr}\,\hat{\mathrm{id}}_A = \delta(e)\cdot|G|$, and so this already cancels the factor $|G|$ in the result. The additional factor $1/\delta(e)$ compensates the fact that $\mathsf{tr}\left|g^{-1}\right\rangle\!\left\langle g^{-1}\right| = \delta(e)$.

construct an example using pure states. Let therefore $\hat{\varsigma}_S = |\psi\rangle\langle\psi|_S$ be a pure system state, and let $\hat{\sigma}_A = |\phi\rangle\langle\phi|_A$ be pure and such that $\mathsf{G}_A[|\phi\rangle\langle\phi|_A] = |\phi\rangle\langle\phi|_A$. The second requirement can only be achieved if G acts in the trivial representation on $|\phi\rangle_A$. As is explained in appendix B.7, a subspace of $L^2(G)$ carrying the trivial representation is only guaranteed to exist if G is compact; and for the *centrally extended Galilei group* (which is not compact), there is none, as will become clear in section 5.4. In general, such a $|\phi\rangle_A$ will thus not exist in $L^2(G)$, but it does in the larger space $\overline{L^2(G)}$ [13].

Such trivial representation spaces in the larger space of a rigged Hilbert space construction are a central object of study in the so-called *perspective-neutral* approach to quantum reference frames [9–14], similar to spaces found in the *Page-Wootters formalism* for time evolution [28, 29]. The perspective-neutral approach builds on pure states, and thus roughly speaking needs a notion of G-invariant pure states, leading to the consideration of subspaces which transform trivially under G.

When working with G-invariant pure states, it makes sense to introduce the *coherent G-twirl* (see [13], above sources, and [49]):

$$\hat{G} := \int_G \hat{U}(g), \tag{3.62}$$

where $\hat{U}$ is the representation on the Hilbert space in question.[8] The coherent G-twirl has similar properties to our (incoherent) G-twirl: it satisfies $\hat{G}^2 \propto \hat{G}$, and it holds that

$$\hat{L}(g)\hat{G}\,|\psi\rangle = \hat{G}\,|\psi\rangle\,, \qquad \forall\, g \in G. \tag{3.63}$$

i.e. $\hat{G}$ acts proportional to the identity on *coherently G-invariant states*, and $\hat{G}\,|\psi\rangle$ transforms in the trivial representation of G, even if this representation is not necessarily a subspace of $L^2(G)$ for non-compact groups. The important difference to our G-twirl G is that $\hat{G}$ preserves the purity of states. Finally, coherently G-invariant pure states are also G-invariant in our sense, but the converse is not true.

If we now take $|\phi\rangle_A$ such as to satisfy $\hat{G}_A\,|\phi\rangle_A \propto |\phi\rangle_A$, we can fulfil the above requirements for the state Alice sees. Bob's state (3.59) is now

$$\begin{aligned}\hat{\rho}_{ABS|B} = \mathsf{U}^\dagger_{A\to B}[\hat{\rho}_{ABS|B}] = \int_G &\mathrm{d}g_1\,\mathrm{d}g_2\;\langle g_1|\varphi\rangle\langle\varphi|g_2\rangle\,\times \\ &\times\,|g_1^{-1}\rangle\langle g_2^{-1}|_A \otimes |\phi\rangle\langle\phi|_B \otimes \hat{U}^\dagger_S(g_1)\;|\psi\rangle\langle\psi|_S\;\hat{U}_S(g_2),\end{aligned} \tag{3.64}$$

If $|\varphi\rangle$ is in a superposition of classical reference frame states, then this is entangled, as we wanted to show. To completely make contact with the relativity of entanglement in [8], we trace out B, yielding:

$$\begin{aligned}\hat{\rho}_{BS|B} = \mathrm{tr}\,|\phi\rangle\langle\phi|_B \int_G &\mathrm{d}g_1\,\mathrm{d}g_2\;\langle g_1|\varphi\rangle\langle\varphi|g_2\rangle\,\times \\ &\times\Big(\,|g_1^{-1}\rangle_A\,\hat{U}^\dagger_S(g_1)\,|\psi\rangle_S\,\Big)\Big(\,\langle g_2^{-1}|_A\,\langle\psi|_S\,\hat{U}_S(g_2)\Big),\end{aligned} \tag{3.65}$$

which is also clearly entangled if $|\varphi\rangle$ is in a superposition. There are other examples of relative entanglement demonstrated in [8]; we will however not consider them here.

Relativity of Entanglement in General? We have seen that the simplest example of entanglement relativity among those described in [8] does not generally occur in the approach [15] considered here. More precisely, we were able to reproduce entanglement

[8] In contrast to the G-twirl (3.26) we do not divide by $|G|$ here. This is done mostly to match the usual definition (see *e.g.* [13]).

relativity if Alice's own state was taken to transform trivially under G. In hindsight, this is not surprising, since assuming a trivial transformation is equivalent to not including Alice's own state at all, as is done in [8] (recall also example 3.5). Entanglement relativity did not occur if a state of maximal entropy was taken for Alice's own state.

The different outcomes (*i.e.* presence or absence of entanglement) could potentially be experimentally distinguished, and so the two extremes are *not equivalent*. This raises the question of which way of dealing with Alice's own state is correct: can she observe internal, G-invariant degrees of freedom of herself as we assumed, or can she not observe herself at all?

What is certain, is that our approach is able to model both possibilities, being general enough. And if we allow internal degrees of freedom, then whether relativity of entanglement occurs seems to depend on Alice's state of knowledge about her internal degrees of freedom. Finally, it should be mentioned that besides relativity of entanglement, one can also consider the related relativity of *partitioning of observable algebras* described in [15]; it can be a more powerful tool to gain insight into the structure of reference frame transformations than relativity of entanglement.

3.5 Coherent, Perspective-Neutral Special Case

We saw in the previous section how employing methods from the coherent perspective-neutral approach can help us reproduce relativity of entanglement. Here we briefly explore how our approach allows for a simple implementation of a coherent perspective-neutral formalism as a special case. But of course, as we argued in the introduction, such a formalism will be incompatible with information theory of imperfect reference frames (*e.g.* [19]).

Jumps. Let $|\psi\rangle_{ABS}$ be coherently G-invariant in the external view, *i.e.* such that

$$\hat{L}_A(g) \otimes \hat{L}_B(g) \otimes \hat{U}_S(g)\,|\psi\rangle_{ABS} = |\psi\rangle_{ABS}\,, \qquad \forall\, g \in G. \tag{3.66}$$

Writing $\langle g| = \langle e|\,\hat{L}^\dagger(g)$, we can compute the jump:

$$\begin{aligned}\hat{U}^\dagger_{\to A}\,|\psi\rangle_{ABS} &= \int_G \mathrm{d}g\,\big(\,|g\rangle\langle e|_A \otimes \hat{\mathrm{id}}_{BS}\big)\,\big(\hat{L}^\dagger_A(g) \otimes \hat{L}^\dagger_B(g) \otimes \hat{U}^\dagger_S(g)\big)\,|\psi\rangle_{ABS} \\ &= \int_G \mathrm{d}g\ |g\rangle\langle e|_A\,|\psi\rangle_{ABS}\,. \end{aligned}\tag{3.67}$$

In other words, the jump into Alice's frame is obtained simply by *conditioning* $|\psi\rangle_{ABS}$ with $\langle e|_A$ on A and setting the state on A to the G-invariant state $\int_G \mathrm{d}g\ |g\rangle_A$. The conditioning becomes even more apparent if we also trace out A:

$$\mathsf{tr}_A \circ \mathsf{U}^\dagger_{\to A}\big[\,|\psi\rangle\langle\psi|_{ABS}\big] = |G| \cdot \langle e|_A\,|\psi\rangle\langle\psi|_{ABS}\,|e\rangle_A\,. \tag{3.68}$$

The factor $|G|$ stems from $\mathsf{tr}\big[\big(\int_G \mathrm{d}g\ |g\rangle\big)\big(\int_G \mathrm{d}g'\ \langle g'|\big)\big] = |G|$.

Conditioning on the jumped-to reference frame is the central mechanism through which jumps from the external view are carried out in perspective-neutral approaches [9–14].

Transformations. This coherent perspective-neutral special case is also capable of meaningful reference frame transformations. For this, take Alice's state to be $|\psi\rangle = |\phi\rangle_B\,|\varphi\rangle_{AS}$ and coherently G-invariant on A. We then find using (3.19) that

$$\hat{U}^\dagger_{A\to B}\,|\psi\rangle_{ABS} = \int_G \mathrm{d}g\ \langle g|\phi\rangle \cdot \big|g^{-1}\big\rangle_A\,\big(\hat{R}^\dagger_B(g) \otimes \hat{U}^\dagger_S(g)\big)\,|\varphi\rangle_{BS}\,. \tag{3.69}$$

The result is typically entangled, with different branches corresponding to different transformations applied to BS, selected according to $|\phi\rangle_B$.

4. Transformations Between Imperfect Quantum Reference Frames

In the last chapter we introduced the unitary transformations between perfect quantum reference frames, eventually making contact with the formalism of [15]. These transformations have been defined with an incoherent notion of G-invariance (*i.e.* we require (3.24) of G-invariant states), and are thus compatible with the rich information theory of imperfect reference frames mentioned in the introduction. Currently, we have achieved two of the three requirements we posed for our formalism in the introduction. The remaining requirement was that our formalism be capable of dealing with imperfect reference frames. In this chapter we will thus take the formalism of previous chapter and extend it to imperfect reference frames.

We begin by noting in section 4.1 that the formalism of last chapter cannot deal with imperfect reference frames without generally compromising unitarity. In section 4.2 we propose to solve this problem by *embedding* imperfect reference frames into perfect reference frames, and argue why this approach also makes sense physically. Section 4.3 is concerned with constructing this embedding and discusses its properties. In section 4.4 we then discuss the view of observers in imperfect frames in light of the embedding and see that transformations between imperfect reference frames truly need the larger, perfect reference frames into which the imperfect frames are embedded, in order to remain unitary. Finally, we discover an important trait of imperfect reference frames in section 4.5: transforming into an imperfect reference frame results in a "fuzzy view" onto the world, even if the reference frame was in a classical reference frame state before the transformation (more precisely, the observed system states are always mixed, centred around the state expected from perfect frames).

4.1 The Problem of Unitarity

The reference frame transformation of [15] which we derived in the previous chapter is

$$\hat{U}^{\dagger}_{A\to B} = \int_G \mathrm{d}g\,\mathrm{d}g' \;\left|g^{-1}\right\rangle\langle g'|_A \otimes |g'g\rangle\langle g|_B \otimes \hat{U}^{\dagger}_S(g). \tag{4.1}$$

Let us try to adapt it to serve also as transformation between imperfect reference frames.

The most straightforward approach would be to replace the classical perfect reference frame states by classical imperfect reference frame states, that is $|g\rangle_R \rightsquigarrow |\Gamma_g\rangle_{\tilde{R}}$; here, R is either A or B, and $\tilde{R}$ is either $\tilde{A}$ or $\tilde{B}$. To distinguish perfect and imperfect reference frames we denote imperfect reference frames with a tilde. Thus, our attempt at a reference frame transformation between imperfect frames would be

$$\hat{U}^{\dagger}_{\tilde{A}\to\tilde{B}} = \int_G \mathrm{d}g\,\mathrm{d}g' \;\left|\Gamma_{g^{-1}}\right\rangle\langle\Gamma_{g'}|_{\tilde{A}} \otimes |\Gamma_{g'g}\rangle\langle\Gamma_g|_{\tilde{B}} \otimes \hat{U}^{\dagger}_S(g). \tag{4.2}$$

This new transformation would act on a pure state $|\psi\rangle_{ABS} = |\phi\rangle_{\tilde{A}} |\varphi\rangle_{\tilde{B}} |\chi\rangle_S$ as

$$\hat{U}^\dagger_{\tilde{A}\to\tilde{B}} |\psi\rangle_{ABS} = \int_G \mathrm{d}g\,\mathrm{d}g'\; \langle\Gamma_{g'}|\phi\rangle \langle\Gamma_g|\varphi\rangle\; \left|\Gamma_{g^{-1}}\right\rangle_{\tilde{A}} |\Gamma_{g'g}\rangle_{\tilde{B}}\, \hat{U}^\dagger_S(g) |\chi\rangle_S \,. \tag{4.3}$$

Unsurprisingly, the transformations which act on S are selected according to the state on B, as was the case for transformations between perfect reference frames. $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ can certainly deal with mixed states, and it can transform between imperfect reference frames. *It is however not unitary.*

To see this, it suffices to construct a counterexample. Let us take a very simple one:

Example 4.1: Counterexample to Unitarity of $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$

Let the rotation $\varphi \in G = \mathrm{U}(1)$ by the angle φ act on $\mathcal{H} = \mathbb{C}$ by multiplication of the phase $\mathrm{e}^{\mathrm{i}\varphi}$. Let the classical reference frame states be $|\Gamma_\theta\rangle := \mathrm{e}^{\mathrm{i}\theta}$. In particular,

$$\langle\Gamma_{\theta'}|\Gamma_\theta\rangle = |\Gamma_\theta\rangle\langle\Gamma_{\theta'}| = \mathrm{e}^{\mathrm{i}(\theta-\theta')}. \tag{4.4}$$

One readily checks that the above representation of G on $\mathbb{C}$ together with these classical reference frame states defines an imperfect reference frame in accord with definition 2.17 (note that the Haar measure is simply $\mathrm{d}\varphi$, and G is compact). Let us further simplify things by considering the system $\mathcal{H}_S = \mathbb{C}$ with G acting trivially.

We then compute, leaving out S since it is trivial:

$$\begin{aligned}
\hat{U}_{\tilde{A}\to\tilde{B}}\hat{U}^\dagger_{\tilde{A}\to\tilde{B}} = &\int_{[0,2\pi)} \mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1'\,\mathrm{d}\theta_2' \times \\
&\times \langle\Gamma_{-\theta_2}|\Gamma_{-\theta_1}\rangle \left\langle\Gamma_{\theta_2'+\theta_2}\middle|\Gamma_{\theta_1'+\theta_1}\right\rangle\, \left|\Gamma_{\theta_2'}\right\rangle\!\left\langle\Gamma_{\theta_1'}\right|_A \otimes |\Gamma_{\theta_2}\rangle\langle\Gamma_{\theta_1}|_B \\
= \int_{[0,2\pi)} &\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1'\,\mathrm{d}\theta_2'\, \exp\left(\mathrm{i}\left[\theta_2 - \theta_1 - \theta_2' - \theta_2 + \theta_1' + \theta_1 + \theta_2' - \theta_1' + \theta_2 - \theta_1\right]\right) \\
&= 4\pi^2 \int_{[0,2\pi)} \mathrm{d}\theta_1\,\mathrm{d}\theta_2\, \exp\left(\mathrm{i}\left[\theta_2 - \theta_1\right]\right) = 0.
\end{aligned} \tag{4.5}$$

Hence, $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ is not unitary.

We thus see that $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ cannot be the solution to our problem. We will turn to another, more successful approach in the next section.

Before doing this, let us pause and ask whether loss of unitarity (and a spectacular one at that) in example 4.1 was only due to the contrived nature of the example. Upon closer inspection we note that θ_1 and θ_2 were not forced to be equal, since the classical reference frame states were not orthogonal. This is true for general imperfect reference frames, and we thus always expect at least the integrals $\int_G \mathrm{d}g_1\,\mathrm{d}g_2$ in the expression for $\hat{U}_{\tilde{A}\to\tilde{B}}\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$. Now as we can infer from $\hat{U}_{\tilde{A}\to\tilde{B}}\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ in example 4.1, this means that $\hat{U}_{\tilde{A}\to\tilde{B}}\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ will generally be a mixture of $|\Gamma_{g_1}\rangle\langle\Gamma_{g_2}|$ in the B part, *for all combinations of* $g_1, g_2 \in G$. The off-diagonal terms will typically prevent the expression from equalling the identity. Note that in our example, off-diagonal and diagonal terms are essentially merged, since $\mathcal{H} = \mathbb{C}$ is one-dimensional. In conclusion, non-unitarity is expected to occur much more generally, and we can confidently abandon the idea of $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$. However, as we will see in section 4.4, $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ still has a physical meaning: it describes what Alice thinks Bob sees based on her own state.

4.2 Embedding of Imperfect Quantum Reference Frames

Our interest in imperfect reference frames was mostly motivated by practical limitations: quantum objects in nature typically satisfy some kind of uncertainty relation, which prevent us from using them as perfect quantum reference frames. For instance, a quantum particle cannot be used as a perfect quantum reference frame for the Galilei group, as we will see in section 6.1.

Our interest in unitarity on the other hand is less inspired by practical aspects and rather by conceptual ideas: our reference frames and systems in a sense should exist in their own right and can simply be "marvelled at" from different perspectives, corresponding to the points of view of our reference frames; this idea is nowhere as clear as in the existence of the external view described in section 3.3. As walking around a sculpture in a museum does not alter the sculpture in any way, we do not want that transforming from one reference frame to another is in any way irreversible.

Now, from certain points of view one might not be able to see the sculpture very clearly. Analogously, we can take the standpoint that in imperfect reference frames, one simply cannot see the rest of nature (observed system and other reference frames) clearly, but that *if one had access to a perfect reference frame, the view would be better.* This gives rise to the idea that imperfect quantum reference frames should in principle be able to be improved, until they become perfect, without at all changing the rest of nature they observe.

We thus propose the idea that any imperfect quantum reference frame, corresponding to the point of view of some physical object, can be understood as part of a larger, perfect quantum reference frame, which is however ideal in the sense that it cannot be fully accessed by the observer due to the limitations of the physical object. This leads us to *embed* physical imperfect quantum reference frames into ideal perfect reference frames through homomorphisms, *i.e.* $\mathcal{H}_{\tilde{R}} \to \mathcal{H}_R$. Again, we denote imperfect frames with a tilde and perfect frames without. The reference frame transformations would then occur between these ideal perfect frames, thus ensuring unitarity. Observers would however not have access to the full Hilbert spaces of the ideal frames, but only to the subspaces corresponding to the embedded imperfect frames.

Interpreting imperfect reference frames as part of perfect reference frames is also interesting from a purely mathematical standpoint, since $L^2(G)$ contains irreducible representations as subspaces (see section 5.4 for details; particularly, if G is compact, then every finite-dimensional irreducible representation of G is contained [41]).[1] One can even show [19] that for compact G, imperfect reference frames are "better" if they are larger subspaces of $L^2(G)$.

4.3 Embedding Construction

We now turn to constructing an embedding of an imperfect quantum reference frame with Hilbert space $\mathcal{H}_{\tilde{R}}$ into a perfect quantum reference frame with Hilbert space $\mathcal{H}_R$. This embedding should conserve probabilities, hence it should be an *isometry.* Also, we would like it to be *compatible with the representations of* G on $\mathcal{H}_R$ and $\mathcal{H}_{\tilde{R}}$.

[1] Based on this mathematical fact, the supervisors Esteban Castro-Ruiz and Ladina Hausmann originally devised the idea of embedding imperfect frames into perfect ones, shortly before the start of the thesis. Thus was born the motivation for this work.

Provisionally, we will define an embedding $H_{\tilde{R}} \to \mathcal{H}_R$ to be a homomorphism with these two properties.[2] The following result characterizes homomorphisms $\mathcal{H}_{\tilde{R}} \to \mathcal{H}_R$ with the above two properties:

Proposition 4.2: Characterization of Embeddings

Let $\mathcal{H}_{\tilde{R}}$ be an imperfect quantum reference frame for G with unitary representation $\hat{U}_{\tilde{R}}$, and let $\mathcal{H}_R$ be a perfect quantum reference frame for G with left-regular representation $\hat{L}$. Consider a homomorphism $\hat{E} : \mathcal{H}_{\tilde{R}} \to \mathcal{H}_R$ which

(a) preserves scalar products, *i.e.* is an *isometry*:

$$\langle\psi|\, \hat{E}^\dagger \hat{E}\, |\varphi\rangle = \langle\psi|\varphi\rangle\,, \qquad \forall\, |\psi\rangle\,, |\varphi\rangle \in \mathcal{H}_{\tilde{R}}, \tag{4.6}$$

(b) preserves the representation structures, *i.e.*

$$\hat{E}\hat{U}_{\tilde{R}}(g) = \hat{L}_R(g)\hat{E}, \qquad \forall g \in G. \tag{4.7}$$

Any such $\hat{E}$ is of the form

$$\hat{E} = \hat{E}(\alpha_e) := \int_G \mathrm{d}g\; |g\rangle\langle\alpha_g|\,, \qquad |\alpha_g\rangle = \hat{U}_{\tilde{R}}(g)\,|\alpha_e\rangle\,, \quad \forall\, g \in G, \tag{4.8}$$

where $|\alpha_e\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$ is a not necessarily normalized or even normalizable vector, such that

$$\hat{E}^\dagger(\alpha_e)\hat{E}(\alpha_e) = \hat{\mathrm{id}}. \tag{4.9}$$

(4.9) is a completeness relation, since for every $|\psi\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$ it holds that

$$\hat{E}^\dagger(\psi)\hat{E}(\psi) = \int_G \mathrm{d}g\, \hat{U}_{\tilde{R}}(g)\,|\psi\rangle\langle\psi|\,\hat{U}^\dagger_{\tilde{R}}(g). \tag{4.10}$$

The proof is provided in appendix A.10.

We see that the embedding $\hat{E}$ is completely determined by the choice of a *seed state* $|\alpha_e\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$ such that $\hat{E}^\dagger(\alpha_e)\hat{E}(\alpha_e)$ is the identity. Once an embedding $\hat{E}$ is chosen, starting from an imperfect reference frame state $|\psi\rangle_{\tilde{R}}$ we can now obtain a corresponding perfect reference frame state $|\psi\rangle_R := \hat{E}\,|\psi\rangle_{\tilde{R}}$. The orbit of states $\hat{U}_{\tilde{R}}(g)\,|\alpha_e\rangle$ generated by the seed state intuitively determines which imperfect reference frame states are mapped to roughly what perfect reference frame state. It would thus make sense to have

$$|\alpha_e\rangle = \frac{1}{\sqrt{r}}\,|\Gamma_g\rangle\,, \tag{4.11}$$

where $r > 0$ is the constant stemming from (2.26); *i.e.* we would want to identify the orbit of the seed state with the classical reference frame states.

Formal Infinities. Still, the identification of classical states and embedding orbit states does not always work: it fails if r is formally infinite, because $\hat{E}(\Gamma_e)/\sqrt{r}$ would then no longer be a well-defined homomorphism. But as we have seen while introducing imperfect reference frames in section 2.4, formal infinities are necessary to capture some cases of reference frames. The same is true for embeddings.

[2] Later it will also be necessary to allow for infinite normalizations in the most general case.

Let us thus refine what we mean by an embedding:

Definition 4.3: Embedding

Let $|\Gamma_e\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$ be a not necessarily normalized or even normalizable state and $r > 0$ a formal constant which could be infinite. Then the formal homomorphism

$$\hat{E} := \frac{\hat{E}(\Gamma_e)}{\sqrt{r}} = \frac{1}{\sqrt{r}} \int_G \mathrm{d}g \; |g\rangle\langle\Gamma_g| \,, \qquad |\Gamma_g\rangle = \hat{U}_{\tilde{R}}(g)\, |\Gamma_e\rangle \,, \quad \forall\, g \in G \tag{4.12}$$

defines a *embedding* of the imperfect quantum reference frame $\mathcal{H}_{\tilde{R}} \subset \bar{\mathcal{H}}_{\tilde{R}}$ into the perfect quantum reference frame $\mathcal{H}_R \subset \bar{\mathcal{H}}_R$ if

$$\frac{\hat{E}^\dagger(\Gamma_e)\hat{E}(\Gamma_e)}{r} = \hat{\mathrm{id}} \tag{4.13}$$

is formally satisfied.

If $r > 0$ is finite and hence $\hat{E}$ is a well-defined operator, then we say that $\hat{E}$ is a *non-formal embedding*. If $r > 0$ is instead formally infinite and hence $\hat{E}$ is only formally defined, then we say that $\hat{E}$ is a *formal embedding*.

Formal embeddings still formally satisfy the two properties (a) and (b) in proposition 4.2, allowing a formally infinite r is thus reasonable. While non-formal embeddings are easier to work with, we are sometimes forced to consider formal ones. This will for instance be the case for embedding imperfect reference frames of the one-dimensional Galilei group in chapter 6. Recall that if G is compact, then the integral in (2.26) always converges, and thus (4.13) holds with a real constant r. For compact G, embeddings are always non-formal.

Seed States in Irreducible Representations. Similarly to proposition 2.21, we can use Schur's lemma [30] to derive the following result:

Proposition 4.4: Seed States in Irreducible Representations

If $\hat{U}_{\tilde{R}}$ is irreducible on $\mathcal{H}_{\tilde{R}}$ then every $|\psi\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$, for which $\hat{E}^\dagger(\psi)\hat{E}(\psi)/r$ is well-defined with $r > 0$ a possibly infinite formal constant, can be appropriately scaled such as to become a valid seed state for an embedding.

This will be useful for the case of the Galilei group in chapter 5, since the representations we will be interested in are irreducible.

Applying the Embedding. With the embedding defined we can now use it to interpret imperfect reference frame states as perfect reference frame states:

Definition 4.5: Embedding of Reference Frame States

Given the state $\hat{\rho}_{\tilde{R}} : \mathcal{H}_{\tilde{R}} \to \bar{\mathcal{H}}_{\tilde{R}}$ of an imperfect reference frame, we obtain the corresponding state $\hat{\rho}_R : \mathcal{H}_R \to \bar{\mathcal{H}}_R$ of the perfect reference frame through the embedding:

$$\hat{\rho}_R := \mathsf{E}[\,\hat{\rho}_{\tilde{R}}], \qquad \mathsf{E}[\,\cdot\,] := \hat{E}\,[\,\cdot\,]\,\hat{E}^\dagger. \tag{4.14}$$

For pure states $|\psi\rangle_{\tilde{R}} \in \bar{\mathcal{H}}_{\tilde{R}}$, the embedded state $|\psi\rangle_R \in \bar{\mathcal{H}}_R$ is

$$|\psi\rangle_R := \hat{E}\,|\psi\rangle_{\tilde{R}} \,. \tag{4.15}$$

When dealing with multiple imperfect reference frames $\mathcal{H}_{\tilde{A}}$, $\mathcal{H}_{\tilde{B}}$, *etc.* and the corresponding perfect frames $\mathcal{H}_A$, $\mathcal{H}_B$, *etc.*, we will distinguish the embeddings acting on different frames by writing E_A, E_B, *etc.* and $\hat{E}_A$, $\hat{E}_B$, *etc.*

For later it will also be useful to introduce the "un-embedding" superoperators $\mathsf{E}^\dagger[\,\cdot\,] := \hat{E}^\dagger\,[\,\cdot\,]\,\hat{E}$, and similarly for $\mathsf{E}^\dagger_A$, $\mathsf{E}^\dagger_B$, *etc.*

If the embedding $\hat{E}$ is non-formal, and $|\psi\rangle_{\tilde{R}} \in \mathcal{H}_{\tilde{R}}$, then the embedded state will also be a proper state: $\hat{E}\,|\psi\rangle_{\tilde{R}} \in \mathcal{H}_R$. If $\hat{E}$ is however formal, then one typically finds that $\sqrt{r}\hat{E}\,|\psi\rangle_{\tilde{R}} \in \bar{\mathcal{H}}_R$, and so the embedded state $\hat{E}\,|\psi\rangle_{\tilde{R}}$ is not even in the larger Hilbert space. To illustrate this, consider the most extreme example:

Example 4.6: Formal Embeddings Require Very Large Vector Spaces

Let G be non-compact and take $\mathcal{H}_{\tilde{R}} \cong \mathbb{C}$ to transform trivially under G. Then, $|\Gamma_g\rangle = |\Gamma_e\rangle$ for all $g \in G$. And hence for $|\psi\rangle \in \mathcal{H}_{\tilde{R}}$,

$$\hat{E}\,|\psi\rangle = \frac{\langle\Gamma_e|\psi\rangle}{\sqrt{r}} \int_G |g\rangle \tag{4.16}$$

is proportional to the constant wave function $\int_G |g\rangle$ on G, which is an element of $\bar{\mathcal{H}}_R$. If $G = (\mathbb{R}, +)$, this constant wave function would be a plane wave with zero momentum. To make $\hat{E}$ a formal isometry however, one requires $r = |G|$, which makes $\hat{E}\,|\psi\rangle$ an infinitely scaled version of $\int_G \mathrm{d}\,|g\rangle$, a state which is not contained in $\bar{\mathcal{H}}_R$.

As before, we will deal with this problem by also allowing vectors in $\bar{\mathcal{H}}_R$ which have been scaled by formally infinite constants; see appendix B.3.

4.4 Observers in Imperfect Reference Frames

After having defined the embedding of imperfect reference frames, we now turn to the points of view of observers in imperfect frames.

Alice & Bob. In contrast to chapter 3, Alice and Bob (and other observers if needed) are now no longer described by perfect quantum reference frames, but by the imperfect quantum reference frames $\mathcal{H}_{\tilde{A}}$ and $\mathcal{H}_{\tilde{B}}$. These imperfect frames are considered part of perfect frames $\mathcal{H}_A$ and $\mathcal{H}_B$ through the embeddings E_A and B_B, according to our idea in section 4.2.

There we also stated that Alice and Bob are assumed to now have access only to the space $\mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_S$, and not the larger space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_S$. Their own states on $\mathcal{H}_{\tilde{A}}$ and $\mathcal{H}_{\tilde{B}}$ respectively will have restrictions, since the embeddings of these states on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively must be G-invariant according to the framework in chapter 3. We will see further down that these restrictions mean that their own states must again be G-invariant, except that we now take the representations $\hat{U}_{\tilde{A}}$ and $\hat{U}_{\tilde{B}}$ respectively in the G-twirl.

Reference Frame State Seen by an Observer. Focus now on one specific observer, say Alice. Let $\mathcal{H}_{\tilde{R}}$ be an imperfect reference frame observed by Alice, and let $\mathcal{H}_R$ be the perfect frame into which $\mathcal{H}_{\tilde{R}}$ is embedded through E. $\mathcal{H}_{\tilde{R}}$ could be Alice's or Bob's frame.

A general state $\hat{\rho}_R$ on of the perfect frame is not necessarily an embedded imperfect state, since $\hat{E}$ is merely an isometry, not an isomorphism:

Proposition 4.7: Characterization of Embedded Imperfect States

If $\hat{E}$ is a non-formal embedding, then

$$\hat{E}\hat{E}^\dagger = \hat{P}_{\mathrm{im}(\hat{E})} \tag{4.17}$$

is the orthogonal projector onto the image $\mathrm{im}(\hat{E}) \subset \mathcal{H}_R$ of $\hat{E}$.

If $\hat{E}$ is an informal embedding, then it still holds that $\hat{E}\hat{E}^\dagger$ acts as the identity on the image $\mathrm{im}(\hat{E})$ (which now also contains infinitely scaled vectors).

Consequently, for a general embedding $\hat{E}$, a state $\hat{\rho}_R$ of the perfect reference frame is *not* an embedded imperfect state if

$$\mathsf{E} \circ \mathsf{E}^\dagger[\hat{\rho}_R] \neq \hat{\rho}_R. \tag{4.18}$$

Furthermore, if $\hat{E}$ is non-formal, then the converse is also true: $\hat{\rho}_R$ is an embedded imperfect state if and only if $\mathsf{E} \circ \mathsf{E}^\dagger[\hat{\rho}_R] = \hat{\rho}_R$.

We show this in appendix A.11.

So if a general state $\hat{\rho}_R$ of the perfect frame may not be an embedded imperfect state, which is the state observed by Alice? We take the standpoint that she should only see those aspects of $\hat{\rho}_R$ which can be understood in terms of embedded imperfect states. We implement this by defining Alice's observed state to be the "un-embedded" state

$$\hat{\rho}^u_{\tilde{R}} := \mathsf{E}^\dagger[\hat{\rho}_R]. \tag{4.19}$$

If $\hat{\rho}_R = \mathsf{E}[\hat{\sigma}_{\tilde{R}}]$ is an embedded imperfect state, then $\hat{\rho}^u_{\tilde{R}} = \mathsf{E}^\dagger \circ \mathsf{E}[\hat{\sigma}_{\tilde{R}}] = \hat{\sigma}_{\tilde{R}}$, as reasonably expected. If $\hat{\rho}_R$ is more general, then proposition 4.7 (at least for non-formal embeddings) shows that $\hat{\rho}^u_{\tilde{R}} = \mathsf{E}^\dagger[\hat{\rho}_R] = \mathsf{E}^\dagger \circ \mathsf{E} \circ \mathsf{E}^\dagger[\hat{\rho}_R]$ projects onto the image of E and then un-embeds the resulting state. Thus, (4.19) is a reasonable definition for the state observed by Alice.

Note that if $\hat{\rho}_R$ is in the kernel of $\mathsf{E}^\dagger$, then $\hat{\rho}^u_{\tilde{R}} = 0$ and Alice observes no state at all. Physically this simply means that $\hat{\rho}_R$ contains no information whatsoever which could be stored in an imperfect state. Thus, in that case it makes sense to *define* $\hat{\rho}^u_{\tilde{R}}$ to be the state of minimal information [45], *i.e.* $\hat{\rho}^u_{\tilde{R}} := \hat{\mathrm{id}}_{\tilde{R}}$ up to normalization. To summarize:

Definition 4.8: Imperfect States Seen By Observers

Let $\hat{\rho}_R$ be the state of a perfect reference frame, into which an imperfect reference frame $\mathcal{H}_{\tilde{R}}$ is embedded through the embedding E. An observer observing $\mathcal{H}_{\tilde{R}}$ (not necessarily the observer associated with the frame) then has access to the un-embedded state (up to normalization)

$$\hat{\rho}^u_{\tilde{R}} := \begin{cases} \mathsf{E}^\dagger[\hat{\rho}_R] & \text{if } \mathsf{E}^\dagger[\hat{\rho}_R] \neq 0, \\ \hat{\mathrm{id}}_{\tilde{R}} & \text{otherwise.} \end{cases} \tag{4.20}$$

Assume now that $\mathcal{H}_{\tilde{R}}$ is Alice's own reference frame. Since $\hat{\rho}_R$ must then be G-invariant, it follows that $\hat{\rho}^u_{\tilde{R}}$ must also be G-invariant:

Proposition 4.9

It holds that

$$\mathsf{G} \circ \mathsf{E}^\dagger = \mathsf{E}^\dagger \circ \mathsf{G}. \tag{4.21}$$

If R is an observer's own frame and hence $\mathsf{G}[\hat{\rho}_R] = \hat{\rho}_R$, then the un-embedded state $\hat{\rho}^u_{\tilde{R}}$ of the observer's own frame must be G-invariant as well.

Proof. The condition (4.7) implies $\hat{E}^\dagger \hat{L}_R(g) = \hat{U}(g)\hat{E}^\dagger$, showing (4.21). Thus, if $\mathsf{G}[\hat{\rho}_R] = \hat{\rho}_R$, then $\mathsf{G}[\hat{\rho}^u_{\tilde{R}}] = \mathsf{G} \circ \mathsf{E}^\dagger[\hat{\rho}_R] = \mathsf{E}^\dagger \circ \mathsf{G}[\hat{\rho}_R] = \mathsf{E}^\dagger[\hat{\rho}_R] = \hat{\rho}^u_{\tilde{R}}$. □

Observables. An observer may wish to measure an observable $\hat{O}_{\tilde{R}}$ on their un-embdedded state. This measurement can alternatively be understood as measuring the "embedded observable" $\hat{E}\hat{O}_{\tilde{R}}\hat{E}^\dagger$, a consequence of the cyclicity of the trace:

Proposition 4.10

It holds that

$$\mathsf{tr}\big(\hat{O}_{\tilde{R}}\hat{E}^\dagger\hat{\rho}_R\hat{E}\big) = \mathsf{tr}\big(\hat{E}\hat{O}_{\tilde{R}}\hat{E}^\dagger\hat{\rho}_R\big). \tag{4.22}$$

More generally, let $\hat{A}_{\tilde{R}}$ and $\hat{B}_R$ be operators on $\mathcal{H}_{\tilde{R}}$ and $\mathcal{H}_R$ respectively; it then holds that

$$\mathsf{tr}\big(\hat{E}\hat{A}_{\tilde{R}}\hat{E}^\dagger\hat{B}_R\big) = \mathsf{tr}\big(\hat{A}_{\tilde{R}}\hat{E}^\dagger\hat{B}_R\hat{E}\big). \tag{4.23}$$

This provides another interpretation of the observer's view: the observer can instead access the *perfect state* $\hat{\rho}_R$, but only through measuring *imperfect observables* $\hat{O}_{\tilde{R}}$, which conveniently have no access to those degrees of freedom in $\hat{\rho}_R$ not replicable by embedding an imperfect state.

Jumping and Transforming into Imperfect Frames. Let us consider the situation where an imperfect state is prepared either in the external view or the view of an observer, before we jump or transform into the frame of another observer. For this, we compute $\hat{U}^\dagger_{\to A}\hat{E}_{AB}$ (for jumps) and $\hat{U}^\dagger_{A\to B}\hat{E}_{AB}$ (for transformations), where $\hat{E}_{AB} = \hat{E}_A \otimes \hat{E}_B$. We denote the potentially infinite constants occurring in the embeddings by r_A and r_B respectively.

The former is

$$\hat{U}^\dagger_{\to A}\hat{E}_{AB} = \int_G \mathrm{d}g\ |g\rangle\langle g|_A\, \hat{E}_A \otimes \hat{L}^\dagger_B(g)\hat{E}_B \otimes \hat{U}^\dagger_S(g)$$
$$= \frac{\hat{E}_B}{\sqrt{r_A}} \int_G \mathrm{d}g\ |g\rangle_A \langle\Gamma_g|_{\tilde{A}} \otimes \hat{U}^\dagger_{\tilde{B}}(g) \otimes \hat{U}^\dagger_S(g). \tag{4.24}$$

We have used (4.7) to pull out $\hat{E}_B$ from the integral. Importantly, we could *not* pull out $\hat{E}_A$, since for $g \in G$ there is generally no state $|\psi_g\rangle \in \bar{\mathcal{H}}_{\tilde{A}}$ with the property that $|g\rangle\langle\Gamma_g| = \hat{E}\,|\psi_g\rangle\langle\Gamma_g|$, since such a state would have to overlap $|\Gamma_g\rangle$ but be orthogonal to $|\Gamma_{g'}\rangle$ for $g \neq g' \in G$; this is impossible, since the states $\{|\Gamma_g\rangle\}_{g\in G}$ are not pairwise orthogonal. We similarly find

$$\hat{U}^\dagger_{A\to B}\hat{E}_{AB} = \frac{1}{\sqrt{r_A r_B}} \int_G \mathrm{d}g\,\mathrm{d}g'\ \big|g^{-1}\big\rangle_A \langle\Gamma_{g'}|_{\tilde{A}} \otimes |g'g\rangle_B \langle\Gamma_g|_{\tilde{B}} \otimes \hat{U}^\dagger_S(g). \tag{4.25}$$

This time, we could not pull out any embeddings from the integral, for the same reason.

Let us check whether an initially embedded imperfect is still embedded after the jump or transformation. We can do this by employing the criterion in proposition 4.7. For jumps, we know from (4.24) that the reference frame not jumped-to, *i.e.* B in our case, will be embedded (since $\hat{E}_B$ could be pulled out), so we focus only on the jumped-to frame:

$$\begin{aligned}\hat{E}_A \hat{E}_A^\dagger \hat{U}_{\to A}^\dagger \hat{E}_{AB} &= \hat{E}_A \hat{E}_A^\dagger \frac{\hat{E}_B}{\sqrt{r_A}} \int_G \mathrm{d}g \; |g\rangle_A \langle \Gamma_g|_{\tilde{A}} \otimes \hat{U}_{\tilde{B}}^\dagger(g) \otimes \hat{U}_S^\dagger(g) \\ &= \frac{\hat{E}_B}{\sqrt{r_A}} \int_G \mathrm{d}g \, \mathrm{d}g' \; |g'\rangle_A \langle \Gamma_{g'}|\Gamma_g\rangle \langle \Gamma_g|_{\tilde{A}} \otimes \hat{U}_{\tilde{B}}^\dagger(g) \otimes \hat{U}_S^\dagger(g) \end{aligned} \tag{4.26}$$

Because $\langle \Gamma_{g'}|\Gamma_g\rangle_{\tilde{A}} \not\propto \delta(g'^{-1}g)$, this is not equal to (4.24).[3] Thus, states will generally not be embedded states after having performed a jump from the external view. It is easy to see that the same is true for jumps to external view, the only change being the removal of "†" from the $\tilde{B}$- and S-parts of the expression. With a bit more work, but still with analogous methods, one shows that generally

$$\hat{E}_{AB} \hat{E}_{AB}^\dagger \hat{U}_{A\to B}^\dagger \hat{E}_{AB} \neq \hat{U}_{A\to B}^\dagger \hat{E}_{AB}, \tag{4.27}$$

and of course similarly for the inverse transformation. In conclusion, we have shown the following:

Theorem 4.11: Jumps and Transformations need Perfect Frames

Jumps and reference frame transformations applied to embedded imperfect reference frame states will generally not result in embedded imperfect reference frame states.

Intuitively, the additional resources provided by the perfect frames which were not already there in the imperfect frames, are necessary for jumps and reference frame transformations. This makes sense, since otherwise the embedding would not have been necessary for unitary transformations and our first idea of section 4.1 should have worked.

We stated in the introduction that we were not so much interested in asking how one observer thinks the view of another should look, but rather how different observer views can be transformed to if we assume that they are part of a common reality. Now as it turns out, our framework, geared to answer the second question, can also answer the first. To see this, consider Alice who only knowing the imperfect state $\hat{\rho}^u_{\tilde{A}\tilde{B}S|A}$ available to her wishes to *guess* Bob's state $\hat{\rho}^u_{\tilde{A}\tilde{B}S|B}$ as well as she can. One guess would be for her to assume that the perfect state $\hat{\rho}_{ABS|A}$ is actually the embedding of $\hat{\rho}^u_{\tilde{A}\tilde{B}S|A}$. Under this assumption, Bob's imperfect un-embedded state would be obtained from her imperfect state through

$$\hat{E}_{AB}^\dagger \hat{U}_{A\to B}^\dagger \hat{E}_{AB} = \frac{1}{r_A r_B} \int_G \mathrm{d}g \, \mathrm{d}g' \; |\Gamma_{g^{-1}}\rangle\langle \Gamma_{g'}|_{\tilde{A}} \otimes |\Gamma_{g'g}\rangle\langle \Gamma_g|_{\tilde{B}} \otimes \hat{U}_S^\dagger(g). \tag{4.28}$$

But of course, generally $\hat{\rho}_{ABS|A} \neq \mathsf{E}[\hat{\rho}^u_{\tilde{A}\tilde{B}S|A}]$ and so her guess might not be correct. Note that (4.28) up to a prefactor equals the transformation $\hat{U}^\dagger_{\tilde{A}\to\tilde{B}}$ defined in (4.2) which we considered in section 4.1. What started out as a failed idea for a unitary quantum reference transformation became useful again in a slightly different context, as an answer to what Alice thinks Bob sees.

[3] To see this more clearly, we can *e.g.* condition with $\langle g|_A$ from the left.

4.5 Mixture of Transformations: "Fuzzy View" as a Sign of Imperfection

Consider the embedded imperfect state $\hat{\rho}_{ABS} = \mathsf{E}_A[|\phi\rangle\langle\phi|_{\tilde{A}}]\otimes\mathsf{E}_B[\hat{\sigma}_{\tilde{B}}]\otimes\hat{\varsigma}_S$. Using proposition 3.17, the system state observed by Alice after a jump or transformation starting in the state $\hat{\rho}_{ABS}$ is

$$\begin{aligned}\hat{\rho}_{S|A} = \mathsf{tr}_{AB}\big(\mathsf{U}^{\dagger}_{\to A}[\hat{\rho}_{ABS}]\big) = \mathsf{tr}_{AB}\big(\mathsf{U}^{\dagger}_{B\to A}[\hat{\rho}_{ABS}]\big)\\ = \frac{\mathsf{tr}(\hat{\sigma}_{\tilde{B}})}{r_A}\int_G \mathrm{d}g\,|\,\langle\Gamma_g|\phi\rangle\,|^2\,\hat{U}^{\dagger}_S(g)\hat{\varsigma}_S\hat{U}_S(g).\end{aligned} \tag{4.29}$$

To simplify the trace in front, we have used proposition 4.10. Similarly to example 3.18, Alice observes a mixture of system states: each term in the mixture is a transformed version of $\hat{\varsigma}_S$, with transformations selected according to $|\,\langle\Gamma_g|\phi\rangle\,|^2$.

In contrast to the case of perfect reference frames in example 3.18, mixtures of transformations now occur even if $|\phi\rangle = |\Gamma_{g_0}\rangle$ for $g_0 \in G$ is a classical reference frame state, because the classical states are not pairwise orthogonal. When switching into an imperfect reference frame, there will *always* be a mixture of transformations applied to S. This "fuzzy view" of S is a unique feature of imperfect quantum reference frames. We will later illustrate it using the Galilei group in chapter 6.4; here, we investigate the phenomenon generally.

Setup. In what follows, we will often use the abbreviation

$$E(g) := \frac{\langle\Gamma_g|\phi\rangle}{\sqrt{r_A\cdot\langle\phi|\phi\rangle}}, \qquad \forall\, g\in G, \tag{4.30}$$

keeping in mind that $E(g)$ depends not just on $g \in G$, but also on the imperfect reference frame state $|\phi\rangle$ and on the seed state $|\Gamma_e\rangle$ chosen for the embedding. The factors in $E(g)$ are chosen such as to make $|E(g)|^2$ a *probability distribution* over G: from the completeness relation (4.13) and (4.10) it follows that

$$\int_G \mathrm{d}g\,|E(g)|^2 = \langle\phi|\,\frac{\int_G \mathrm{d}g\,|\Gamma_g\rangle\langle\Gamma_g|}{r_A\cdot\langle\phi|\phi\rangle}\,|\phi\rangle = \frac{\langle\phi|\phi\rangle}{\langle\phi|\phi\rangle} = 1. \tag{4.31}$$

$|E(g)|^2$ can be seen as the probability density of measuring the $|\Gamma_g\rangle$ among all other classical reference frame states when $|\phi\rangle$ was prepared.

Entropy. A way to measure the amount of mixture in $\hat{\rho}_{S|A}$ is the *von Neumann entropy.* For a *normalized* state $\hat{\sigma}$ of a quantum system described by a Hilbert space $\mathcal{H}_Q$ is defined as [45][4]

$$H(\hat{\sigma}) := -\mathsf{tr}\big(\hat{\sigma}\log_2\hat{\sigma}\big), \tag{4.32}$$

where the base-2 logarithm is understood in terms of eigenvalues; one uses the convention that $0\cdot log_2 0 = 0$. It holds that $H(\hat{\sigma}) = 0$ if and only if $\hat{\sigma}$ is pure, and

$$H(\hat{\sigma}) \le \log_2(\dim\operatorname{supp}\hat{\sigma}) \le \log_2(\dim\mathcal{H}_Q), \tag{4.33}$$

with equality in the first inequality if and only if $\hat{\sigma} = \hat{\mathrm{id}}_{\operatorname{supp}\hat{\sigma}}/\dim\operatorname{supp}\hat{\sigma}$ and equality in the second inequality if and only if $\hat{\sigma} = \hat{\mathrm{id}}/\dim\mathcal{H}_Q$. This follows easily from considering

[4]The standard notation for entropy in quantum information theory is $H(Q)_{\hat{\sigma}}$, where Q labels the quantum system. We will use a simpler notation only involving the state, since the system will always be at least implicitly clear.

an orthonormal which diagonalizes $\hat{\sigma}$; see [45] for a discussion of the finite-dimensional case. The infinite-dimensional case is not much harder, except that now $\log_2(\dim \mathcal{H}_Q)$ and potentially $\log_2(\dim \operatorname{supp} \hat{\sigma})$ must be seen as formal infinities, and $\hat{\mathrm{id}}/\dim \mathcal{H}_Q$ and potentially $\hat{\mathrm{id}}_{\operatorname{supp}\hat{\sigma}}/\dim \operatorname{supp}\hat{\sigma}$ are formally defined states.

Since our states are not necessarily normalized, we will work with a slightly more general expression, see definition B.9 in appendix B.5:

$$H(\hat{\sigma}) := -\mathsf{tr}\left(\frac{\hat{\sigma}}{\mathsf{tr}\,\hat{\sigma}} \log_2 \frac{\hat{\sigma}}{\mathsf{tr}\,\hat{\sigma}}\right). \tag{4.34}$$

Clearly, this reduces to (4.32) if the state is normalized as $\mathsf{tr}\hat{\sigma} = 1$. We briefly discuss the more general expression (4.34) in appendix B.5; there we will see that the more general expression works nicely with the most often encountered non-normalized states such as $|g\rangle$, $g \in G$, superpositions of such states, and mixtures.

Badness Measure. Let now $\hat{\rho}_{ABS} = \mathsf{E}_A[|\Gamma_{g_0}\rangle\langle\Gamma_{g_0}|_{\tilde{A}}] \otimes \mathsf{E}_B[\hat{\sigma}_B] \otimes \hat{\varsigma}_S$ be such that Alice is in a classical reference frame state. Jumping or transforming into her frame gives

$$\hat{\rho}_{S|A} = C \cdot \int_G \mathrm{d}g\, |E(g)|^2\, \hat{U}_S^\dagger(g) \hat{\varsigma}_S \hat{U}_S(g), \qquad C := \mathsf{tr}\big(\hat{\sigma}_B\big) \cdot \langle\Gamma_{g_0}|\Gamma_{g_0}\rangle_{\tilde{A}}\,. \tag{4.35}$$

We wish to measure the amount of mixture induced by the various transformations on S.

We first note that if $\hat{\varsigma}_S$ is itself mixed, then $H(\hat{\rho}_{S|A})$ measures in part the mixture in $\hat{\varsigma}_S$, which is not our intent. Let thus $\hat{\varsigma}_S = |\chi\rangle\langle\chi|_S$ be pure. We then note that if $\hat{U}_S$ is trivial, it follows from (4.35) that $\hat{\rho}_{S|A}$ is pure, and accordingly, $H(\hat{\rho}_{S|A}) = 0$. This is however hardly an interesting case. On the flip side, with fixed state on $\tilde{A}$, the entropy is maximized if the states $\hat{U}_S^\dagger(g)\,|\chi\rangle_S$ are pairwise orthogonal; this is possible with $\mathcal{H}_S = L^2(G)$, $|\chi\rangle_S = |e\rangle$, $\hat{U}_S = \hat{L}_S$ and thus $\langle e|_S\, \hat{U}_S(g) = \langle g^{-1}|$. The trace of $\hat{\rho}_{S|A}$ in this case is

$$\mathsf{tr}\big(\hat{\rho}_{S|A}\big) = C \cdot \int_G \mathrm{d}g\,\mathrm{d}g'\, |E(g)|^2\, \langle g'|_S\, \hat{L}_S^\dagger(g)\, |e\rangle\langle e|_S\, \hat{L}_S(g)\, |g'\rangle_S = C \cdot \delta(e), \tag{4.36}$$

where we have used (4.31). The entropy is then

$$\begin{aligned} H(\hat{\rho}_{S|A}) = -\int_G \mathrm{d}g\,\mathrm{d}g'\,\mathrm{d}g''\, \langle g|_S\, \frac{|E(g')|^2\, |g'^{-1}\rangle\langle g'^{-1}|_S}{\delta(e)} \log_2\left(\frac{|E(g'')|^2}{\delta(e)}\right) |g''^{-1}\rangle\langle g''^{-1}|\, |g\rangle_S \\ = -\int_G \mathrm{d}g\, |E(g)|^2 \log_2 |E(g)|^2 + \log_2 \delta(e). \end{aligned} \tag{4.37}$$

We will not worry much about the infinite second factor, as it only introduces a constant shift; we will however note that if $|E(g)|^2 = \delta(g)$, then the entropy becomes zero, as expected for a pure state, since the first term cancels the second. Let us thus focus on the first term:

$$H(|E|^2) := -\int_G \mathrm{d}g\, |E(g)|^2 \log_2 |E(g)|^2 \tag{4.38}$$

is the so-called *differential entropy* of the probability distribution $|E(g)|^2$ (see *e.g.* [50]). Since it is, up to an infinite constant offset, the maximum entropy of $\hat{\rho}_{S|A}$ one can obtain with a pure state on $\mathcal{H}_S$, we can see it as quantifying the mixture of $\hat{\rho}_{S|A}$ induced by transformations on S in the worst case. It is the measure of mixture we were looking for.

It is not hard to see that $H(|E|^2)$ satisfies the requirements of a badness measure:

Proposition 4.12

$H(|E|^2)$ is a badness measure according to definition 2.23:

$$H(|E|^2) = \int_G \mathrm{d}g\, f\big(|\langle\Gamma_e|\, \hat{U}_{\tilde{A}}(g)\, |\Gamma_e\rangle\,|\big), \tag{4.39}$$

where

$$f(x) := -\frac{x^2}{r\,\langle\Gamma_e|\Gamma_e\rangle} \log_2 \frac{x^2}{r\,\langle\Gamma_e|\Gamma_e\rangle}. \tag{4.40}$$

5. Galilei Transformations in One Dimension

We mentioned in the introduction that quantum reference frames, particularly imperfect ones, are expected to play an important role in quantum gravity [8, 10, 14, 15, 25–27]. In this chapter, we introduce two groups whose quantum reference frames are thought to be of importance in that regard: the group of *Galilei transformations* in one dimension, *i.e.* the group of *translations* and *boosts*. More precisely, we will introduce two groups describing such transformations: the *one-dimensional isochronous Galilei group* Gal as well as its central extension CGal. These groups will then be applied to quantum reference frame transformations in chapter 6.

In section 5.1 we argue for the importance of Galilei transformations in low-energy quantum gravity applications. Section 5.2 then introduces the one-dimensional isochronous Galilei group as well as its *mass-m representations*, describing how massive quantum particles transform under Galilei transformations. We also discuss the action of those representations in phase space, making use of the *Wigner distribution* formalism. While physically very important, the mass-m representations are projective, and one must consider the *centrally extended, one-dimensional isochronous Galilei group* to turn them into non-projective unitary representations, which we do in section 5.3. Section 5.4 then closes with the representation theory of the centrally extended Galilei group.

5.1 Importance of Galilei Transformations

Quantum reference frame transformations can help us understand quantum gravity, for instance by describing the perspective of a mass in superposition. This could help us understand potential superpositions of gravitational fields in the low-energy regime (hence the relevant group is the Galilei group): assuming that the gravitational field of a mass becomes classical in the reference frame of said mass, one could transform into the frame of a mass in superposition, perform calculations there, transform back again and possibly gain valuable insight about the gravitational field of the mass in the original frame where the mass is in a superposition. This idea is illustrated in figure 5.1. Furthermore, we think that the frame of such a mass should be an *imperfect* one, because massive particles cannot be perfectly localized in space, because they have to be also somewhat localized in momentum, which is the case if the energy of the particle is limited (similar restrictions on measurements were already remarked as early as [5]). Thus, we expect to be able to apply our framework of reference frame transformations between imperfect reference frames.

In the context of gravity it is of course interesting to consider quantum reference frames for a group G encoding *spacetime symmetries*: the *Galilei group* of classical relativity, the *Poincaré group* of special relativity [1], or even the *diffeomorphism group* of general relativity [2]. Treating the diffeomorphism group is sadly impossible with our framework, since it is not

Figure 5.1: Left: A gravitating mass in spatial superposition, whose gravitational field strength is measured by a well localized device (say a *Cavendish*-type experimental setup). Without a theory of quantum gravity, we do not know the exact form of the gravitational field of this mass, and we cannot predict what the device measures. Right: When transforming into the frame of the mass, the gravitational field may be assumed to become classical (light purple circles), allowing us to predict the reading of the device. However, the device is no longer localized and accordingly shows a mixture of readings. This thought experiment is again taken up in the outlook, in chapter 7.

a finite-dimensional Lie group. Even discussing the Poincaré group would be too ambitious, since one is forced to consider "mixtures" of space and time (as they commonly occur in special relativity), which requires putting space and time somehow on equal footing. While there are approaches to solve this issue (*e.g.* famously quantum field theory [51]), the problem of time remains a largely open issue (see *e.g.* [52, 53]).

We will thus focus on the classical limit and consider the Galilei group here. For simplicity, we will further consider only a single spatial dimension. Thanks to this, we will not have to deal with rotations, whose quantum reference frames have been studied extensively (see *e.g.* [19]), and wee can focus on *translations* and *boosts* exclusively. For a treatment of the Galilei group in three dimensions, as well as its quantum-mechanical features, see [39].

5.2 Galilei Group

Abstractly, we can define the *one-dimensional Galilei group* as $\mathbb{R} \times \mathbb{R}$ equipped with vector addition:

Definition 5.1: Galilei Group in One Dimension

The *one-dimensional isochronous Galilei group*, or *Galilei group* for short, is defined as [15]

$$\mathrm{Gal}(1) \cong \mathbb{R} \times \mathbb{R}, \qquad (a', v') \cdot (a, v) := (a' + a, v' + v). \tag{5.1}$$

We will abbreviate $\mathrm{Gal} := \mathrm{Gal}(1)$.

Importantly for us, the Haar measure of the Galilei group is particularly simple:

Proposition 5.2: Haar Measure of Gal

Gal is unimodular with left and right Haar measure the Lebesgue measure on $\mathbb{R}^2$:

$$\mathrm{d}g = \mathrm{d}(a, v) := \mathrm{d}a\, \mathrm{d}v. \tag{5.2}$$

We derive this fact in appendix B.6. We always use this normalization of the Haar measure.

Action on a Classical Particle. The physical significance of Gal is best understood by considering the action on a classical particle with position and velocity degrees of freedom (x, u), given by

$$(a, v) : (x, u) \mapsto (a + x, v + u). \tag{5.3}$$

Gal thus acts on the particle through combinations of *translations* controlled by a, and velocity changes, so-called *boosts*, controlled by v. It is readily seen that the particle can be used as a perfect minimal reference frame for Gal provided that all particle states are distinguishable.

Quantum Particle. More interesting for us is the group action on quantum particles, to which we turn now. For an extensive treatment of a single quantum particle, see *e.g.* [30]; we recall here only its most important features. Later in section 6.1 we will see that a single quantum particle cannot provide a perfect reference frame for Gal.

A quantum particle in one dimension is described by the Hilbert space $L^2(\mathbb{R})$. The degrees of freedom x and u have become operators $\hat{x}$ and $\hat{u} = \hat{p}/m$, where $\hat{p}$ is the *momentum operator* and m is the *mass* of the particle. The canonical commutation relations hold:

$$[\hat{x}, \hat{p}] = \mathrm{i} \cdot \hat{\mathrm{id}} \tag{5.4}$$

or equivalently, $[\hat{x}, \hat{u}] = (\mathrm{i}/m) \cdot \hat{\mathrm{id}}$. Without loss of generality we take elements $\psi \in L^2(\mathbb{R})$ to be the *position-space wave functions*, such that the position and momentum operators act as

$$(\hat{x}\psi)(y) = y\psi(y), \qquad (\hat{p}\psi)(y) = -\mathrm{i}\frac{\partial}{\partial y}\psi(y). \tag{5.5}$$

As for $L^2(G)$ we adopt bra-ket notation and often write states $\psi \in L^2(\mathbb{R})$ as $|\psi\rangle$. One can form the improper *position basis* from eigenstates $|x\rangle$ of $\hat{x}$; we normalize them such that

$$\langle x|x'\rangle = \delta(x - x'), \qquad \int_{\mathbb{R}} \mathrm{d}x\, |x\rangle\langle x| = \hat{\mathrm{id}}. \tag{5.6}$$

The improper function described by $|x\rangle$ is the δ-distribution in position-space centred on x, and one can write $\langle x|\psi\rangle = \psi(x)$ for every $\psi \in L^2(\mathbb{R})$.[1] Improper eigenstates $|p\rangle$ of $\hat{p}$, *i.e.* $\hat{p}\,|p\rangle = p\,|p\rangle$, form the improper *momentum basis* of plane waves. We chose the convention

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}px}, \tag{5.7}$$

such that similarly to position eigenstates, we have

$$\langle p|p'\rangle = \delta(p - p'), \qquad \int_{\mathbb{R}} \mathrm{d}p\, |p\rangle\langle p| = \hat{\mathrm{id}}. \tag{5.8}$$

The convention (5.7) also ensures that $\hat{x}$ and $\hat{p}$ are related by a unitary Fourier transform. We will work in the Schrödinger picture and thus see particle states as time-dependent, while observables are not.[2] Often we will however not care about time evolution at all and consider physics on a single time slice by setting $t = 0$.

Action on a Quantum Particle. Gal acts on the quantum particle through the so-called *mass-m representation* [15]:

[1] δ-distributions can also be understood through the method of chapter 2 applied to the Lie group $(\mathbb{R}, +)$.
[2] Except for potentially intrinsic time-dependence.

Definition 5.3: Mass-m Representation of Gal

For $m \in \mathbb{R}$, the *mass-m* representation of Gal acting on $L^2(\mathbb{R})$ is given by

$$\hat{U}_m(a,v) = \exp\left(-\mathrm{i}(a\hat{p} + v\hat{k}_m)\right), \qquad \hat{k}_m := \hat{p}t - m\hat{x}. \tag{5.9}$$

$\hat{p}$ and $\hat{k}_m$ are called the *generators* for translations and boosts respectively. We will sometimes omit the subscript-m's on the representation and the boost generator if the mass is implicitly clear.

Note that $\hat{k}_m$ intrinsically depends on time; this time-dependence is required in order for time evolution of the particle (the Schrödinger equation) to be invariant under the action of $\hat{U}_m(a,v)$ [30].[3] Let us mention the most important features of this representation:

Proposition 5.4: Properties of the Mass Representations of Gal

(a) The generators do not commute:

$$\left[\,\hat{p}, \hat{k}_m\,\right] = \mathrm{i}m \cdot \hat{\mathrm{id}}. \tag{5.10}$$

(b) The representation is unitary projective, and it holds that

$$\hat{U}_m(a',v')\,\hat{U}_m(a,v) = \exp\left(\mathrm{i}\frac{m}{2}(av' - a'v)\right)\hat{U}_m(a'+a, v'+v), \tag{5.11}$$

(c) For $m \neq m'$, the mass-m and mass-m' representations are *inequivalent*, *i.e. there is no* unitary operator $\hat{V}$ such that for all $g \in G$, $\hat{V}\hat{U}_m(g)\hat{V}^\dagger = \hat{U}_{m'}$.

(d) The actions on position and momentum states are

$$\hat{U}_m(a,v)\,|x\rangle = \mathrm{e}^{\mathrm{i}mv\,(x+a/2+vt/2)}\,|x+a+vt\rangle\,, \tag{5.12}$$
$$\hat{U}_m(a,v)\,|p\rangle = \mathrm{e}^{-\mathrm{i}(a+vt)(p+mv/2)}\,|p+mv\rangle\,. \tag{5.13}$$

(e) If $\psi(x)$ is the position-space wave function of a state, then $\hat{U}_m(a,v)$ transforms the wave function as

$$\psi(x) \mapsto \psi(x-a-vt)\,\mathrm{e}^{\mathrm{i}mv\,(x-a/2-vt/2)}. \tag{5.14}$$

If $\tilde{\psi}(p)$ is the momentum-space wave function, then $\hat{U}_m(a,v)$ acts as

$$\tilde{\psi}(p) \mapsto \tilde{\psi}(p-mv)\,\mathrm{e}^{-\mathrm{i}(a+vt)(p-mv/2)} \tag{5.15}$$

(f) For any $0 \neq |\psi\rangle \in L^2(\mathbb{R})$ we have the completeness relation

$$\hat{E}^\dagger(\psi)\hat{E}(\psi) = \int_{\mathrm{Gal}} \mathrm{d}g\,\hat{U}_m(g)\,|\psi\rangle\langle\psi|\,\hat{U}_m^\dagger(g) = \frac{2\pi}{m}\,\langle\psi|\psi\rangle \cdot \hat{\mathrm{id}}. \tag{5.16}$$

Thus, the mass-m representation $\hat{U}$ is *irreducible*, and every such $|\psi\rangle$ can be rescaled into a seed state for an embedding.

The proof is presented in appendix A.12.

We see that momentum eigenstates are boosted into other momentum eigenstates. And

[3] After the proof of proposition 5.4 below, we briefly touch on this subject in appendix A.12.

because momentum eigenstates are orthogonal, we can use them to turn the particle into a perfect reference frame for the subgroup of boosts in Gal. However, since momentum eigenstates are up to a phase insensitive to translations, we cannot extend this perfect reference frame to the full Galilei group (even when ignoring the fact that the representation is projective and hence does not fit definition 2.9). For similar reasons, position eigenstates can be used to make our particle into a perfect reference frame for the subgroup of translations, but they also fail to provide a perfect reference frame for the full group. We will make these ideas precise in section 6.2.

Action in Phase Space. The action of the mass-m representation $\hat{U}_m$ can be illustrated particularly nicely in *phase space* [30, 54–56]. Given a position space wave function ψ of our particle, one introduces the *Wigner quasiprobability distribution* [54]:

$$W_\psi(x,p) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}y\, \psi\left(x - \tfrac{1}{2}y\right) \psi^*\left(x + \tfrac{1}{2}y\right) \mathrm{e}^{\mathrm{i}py}, \tag{5.17}$$

which is real and normalized:

$$\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}p\, W_\psi(x,p) = \langle\psi|\psi\rangle\,. \tag{5.18}$$

W_ψ is a *quasiprobability distribution*, since it can take on negative values on small areas of phase space of the order of 2π [30].[4] The only non-negative Wigner distributions of pure states are those of *squeezed coherent states*, in which case the Wigner distributions are squeezed (*i.e.* not necessarily rotationally symmetric) Gaussian distributions [57]; see also figure 5.2. We will use squeeze coherent states in section 6.3. Importantly, far from every properly normalized function on phase space is a Wigner distribution [58]. The Wigner distribution W_ψ can equivalently be computed from the momentum-space wave function $\tilde{\psi}$:

$$W_\psi(x,p) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, \tilde{\psi}\left(p - \tfrac{1}{2}k\right) \tilde{\psi}^*\left(p + \tfrac{1}{2}k\right) \mathrm{e}^{-\mathrm{i}xk}. \tag{5.19}$$

That both ways of computing the Wigner distribution coincide is a consequence of the fact that for two complex functions ψ, φ with Fourier transforms $\tilde{\psi}$ and $\tilde{\varphi}$,

$$\int_{\mathbb{R}} \mathrm{d}y\, \psi(x - \tfrac{1}{2}y)\varphi^*(x + \tfrac{1}{2}y)\,\mathrm{e}^{\mathrm{i}py} = \int_{\mathbb{R}} \mathrm{d}k\, \tilde{\psi}(p - \tfrac{1}{2}k)\tilde{\varphi}^*(k + \tfrac{1}{2}k)\,\mathrm{e}^{-\mathrm{i}kx}. \tag{5.20}$$

This can be shown with a straightforward computation.

Despite possibly negative parts, one may still think of W_ψ as in some appropriate sense describing the spread of ψ over phase space. This is further accentuated by the following two results [30]: Firstly, if φ is another wave function, then

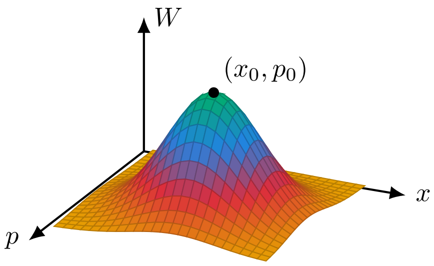
$$|\langle\psi|\varphi\rangle|^2 = 2\pi\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}p\, W_\psi(x,p)\, W_\varphi(x,p). \tag{5.21}$$

Secondly, if $\hat{A}$ is an operator, then one can construct its *Wigner representation* $A(x,p)$ and it holds that

$$\langle\psi|\,\hat{A}\,|\psi\rangle = \int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}p\, W_\psi(x,p)\, A(x,p). \tag{5.22}$$

The Galilei transformation (a,v) acts on W_ψ as a simple phase-space shift:

[4] With units restored, they are of order $2\pi\hbar = h$.



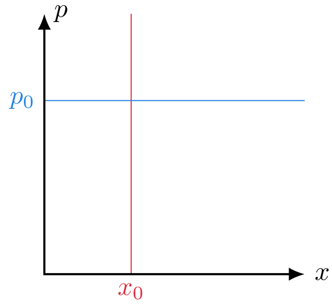


Figure 5.2: Several Wigner distributions. Left: The Wigner distribution of a coherent state with position x_0 and momentum p_0 is a Gaussian distribution centred on (x_0, p_0), and everywhere positive. Right: The Wigner distributions of position and momentum eigenstates have support on vertical and horizontal lines in phase space. Shown are the Wigner distributions for $|x_0\rangle$ (red) and $|p_0\rangle$ (blue). Since overlaps of states can be computed from overlaps of Wigner distributions, and because the Galilei transformation (a, v) shifts the Wigner distribution by (a, v) in phase space, we immediately see that position eigenstates make a perfect reference frame for translations but a terrible one for boosts, while momentum eigenstates do the opposite.

Proposition 5.5: Mass-m Representation in Phase Space

The Galilei transformation (a, v) at $t = 0$ acts on the Wigner distribution W_ψ of a position-space wave function ψ as

$$W_\psi(x, p) \rightsquigarrow W_\psi(x - a, p - mv). \tag{5.23}$$

Proof. According to (5.14) and (5.17), the transformed Wigner distribution is

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}p\,\psi\left(x - a - \tfrac{1}{2}y\right)\psi^*\left(x - a + \tfrac{1}{2}y\right)\mathrm{e}^{\mathrm{i}mv(x-y/2-a/2)}\mathrm{e}^{-\mathrm{i}mv(x+y/2-a/2)}\mathrm{e}^{\mathrm{i}py} \\ = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{d}p\,\psi\left(x - a - \tfrac{1}{2}y\right)\psi^*\left(x - a + \tfrac{1}{2}y\right)\mathrm{e}^{\mathrm{i}(p-mv)y} = W_\psi(x - a, p - mv).\end{aligned} \tag{5.24}$$

□

Finally, it will be useful to compute the Wigner distributions of position and momentum eigenstates:

Example 5.6

The position eigenstate $|x_0\rangle$, $x_0 \in \mathbb{R}$, has the wave function $\delta(x - x_0)$, and hence, using (5.17),

$$W_{x_0}(x, p) = \frac{1}{\pi}\delta(2x - 2x_0)\mathrm{e}^{\mathrm{i}p2(x-x_0)} = \frac{1}{2\pi}\delta(x - x_0). \tag{5.25}$$

The momentum eigenstate $|p_0\rangle$, $p_0 \in \mathbb{R}$, has the wave function $\mathrm{e}^{-\mathrm{i}p_0x}/\sqrt{2\pi}$, and thus

$$W_{p_0} = \frac{1}{4\pi^2}\int_{\mathbb{R}} \mathrm{d}y\,\mathrm{e}^{\mathrm{i}p_0(x-y/2)}\mathrm{e}^{-\mathrm{i}p_0(x+y/2)}\mathrm{e}^{\mathrm{i}py} = \frac{1}{2\pi}\int \mathrm{d}y\,\mathrm{e}^{\mathrm{i}y(p-p_0)} = \frac{1}{2\pi}\delta(p - p_0). \tag{5.26}$$

As expected, position and momentum eigenstates correspond to infinitely high "δ-ridges" in phase space, parallel to the p- and x-axis respectively. This is also shown in figure 5.2.

5.3 Centrally Extended Galilei Group

Because the representation (5.9) of Gal acting on $L^2(\mathbb{R})$ described above is projective rather than linear, the simple tools of linear representation theory do not apply. But as we have already argued in section 2.2, one can consider the *central extension* of our group to turn projective representations into non-projective ones [36].

In our case it is possible to see $\hat{U}_m$ as a non-projective representation of a larger group CGal := CGal(1), the *centrally extended Galilei group*. Furthermore, one can even argue that the physically natural group to consider is not Gal, but CGal: very roughly speaking, a rigorous treatment of *Bargmann superselection rules* [59] for mass, *i.e.* the mechanism required to prevent superpositions of different particle masses in non-relativistic quantum mechanics, requires mass to become a dynamical quantity, which implies the centrally extended Galilei group as the relevant symmetry group [39].

The centrally extended Galilei group CGal is defined as [15]:

Definition 5.7: Centrally Extended Galilei Group in One Dimension

The *centrally extended, one-dimensional isochronous Galilei group*, or *centrally extended Galilei group* for short, is defined as

$$\mathrm{CGal} \cong \mathbb{R}^3, \qquad (\theta', a', v') \cdot (\theta, a, v) := \left(\theta' + \theta + \frac{av' - a'v}{2}, a' + a, v' + v\right). \tag{5.27}$$

To check that this is indeed a group, we must show the associativity of the group multiplication, the existence of an identity element, and the existence of an inverse for each element. Associativity follows from $(av' - a'v) + (a' + a)v'' - a''(v' + v) = (a'v'' - a''v') + a(v' + v'') - v(a' + a'')$ and the associativity of addition. The identity is

$$e = (0, 0, 0), \tag{5.28}$$

and the inverse of (θ, a, v) is

$$(\theta, a, v)^{-1} = (-\theta, -a, -v). \tag{5.29}$$

Compared to Gal, CGal now also contains "θ-translations". We will see that these are required to make the mass-m representations non-projective. Note that CGal is *not Abelian* thanks to the third term in the θ-part of the group multiplication.

Analogously to Gal, we show in appendix B.6 that:

Proposition 5.8: Haar Masure of CGal

CGal is unimodular with left and right Haar measure the Lebesgue measure on $\mathbb{R}^3$:

$$\mathrm{d}g = \mathrm{d}(\theta, a, v) := \mathrm{d}\theta \, \mathrm{d}a \, \mathrm{d}v. \tag{5.30}$$

We will always stick to this normalization of the Haar measure.

Mass-m Representations Recovered. Let us now see why this larger group can capture the irreducible mass-m representations of Gal as unitary (non-projective) representations:

Definition 5.9: Mass-m Representation of CGal

For $m \in \mathbb{R}$, the mass-m representation of CGal acting on $L^2(\mathbb{R})$ is defined as [15]

$$\hat{U}_m(\theta, a, v) := \mathrm{e}^{\mathrm{i}m\theta} \hat{U}_m(a, v), \tag{5.31}$$

where $\hat{U}_m(a, v)$ is the projective representation of Gal encountered in definition 5.3. Again, we sometimes omit the subscript-m if the mass is clear from the context.

The action of $\hat{U}_m(\theta, a, v)$ on different particle states and wave functions readily follows from the action of $\hat{U}_m(a, v)$ derived in proposition 5.4. Also, $\hat{U}_m(0, a, v) = \hat{U}_m(a, v)$, and so the mass-$m$ representation of CGal clearly contains the mass-m representation of Gal. The generators of translations and boosts are still $\hat{p}$ and $\hat{k}_m$ respectively; the generator of θ-translations is the *mass operator* $\hat{m} = m \cdot \hat{\mathrm{id}}$. In this sense, it is possible to see θ as "conjugate" to the mass m. More rigorously, a quantum system which is invariant under θ-translations must conserve the mass m. This is related to the way in which θ-translations naturally occur if one is studying superselection rules for mass m [39].

The irreducibility and inequivalence of the mass representations of Gal carry over to the mass representations of CGal:

Proposition 5.10: Properties of the Mass Representations of CGal

(a) The mass-m representation of CGal is unitary non-projective.

(b) The mass-m representation of CGal is irreducible.

(c) For $m \neq m'$, the mass-m and mass-m' representations of CGal are inequivalent.

(d) For any $|\psi\rangle \in L^2(\mathbb{R})$ it holds that

$$\int_{\mathrm{CGal}} \mathrm{d}g \, \hat{U}_m(g) \, |\psi\rangle \, \langle\psi| \, \hat{U}_m^\dagger(g) = \frac{2\pi \cdot \delta(m)|_0}{m} \, \langle\psi|\psi\rangle \cdot \hat{\mathrm{id}}, \tag{5.32}$$

where

$$\delta(m)|_0 := \delta(m)|_{m=0} = \big(\textstyle\int_{\mathbb{R}} \mathrm{d}\theta \, \mathrm{e}^{\mathrm{i}m\theta} \big)\big|_{m=0} = \textstyle\int_{\mathbb{R}} \mathrm{d}\theta. \tag{5.33}$$

Consequently, any such $|\psi\rangle$ can be rescaled into a valid seed state for an embedding.

We provide the proof in appendix A.13.

Compact Central Extension. The central extension of Gal into CGal further increases the group measure by an infinite factor $\int_{\mathbb{R}} \mathrm{d}\theta$. As we see when comparing (5.32) to (5.16), it makes the difference between a well-defined completeness relation in the case of Gal and one in need of formally infinite normalization in case of CGal. To prevent this infinity, one can alternatively extend Gal by a compact Abelian group, say $U(1)$, instead of by the non-compact Abelian group $(\mathbb{R}, +)$ [36].

The mass-m representations would then still be defined as in definition 5.9, except that θ must now be understood modulo 2π. For $\hat{U}_m$ to then be well-defined however, we need $m \in \mathbb{Z}$. Extending by $U(1)$ has the effect of removing an annoying formal infinity, but it also forces us to quantize the mass. While interesting, we did not investigate this possibility further and instead decided on the central extension described above.

5.4 Representation Theory of the Centrally Extended Galilei Group

We begin this section by noting that CGal is *not compact*, and one can thus not rely on the well-understood representation theory of compact topological groups, which we briefly cover in appendix B.7. Nevertheless, we will see that the representation theory of CGal bears many resemblances to that of compact groups.

Decomposition of $L^2(G)$. Without considering the representation structure for now, it holds that

$$L^2(\mathrm{CGal}) \cong L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong \int_{\mathbb{R}}^{\oplus} \mathrm{d}m \, L^2(\mathbb{R}) \otimes L^2(\mathbb{R})^*. \tag{5.34}$$

This decomposition resembles the decomposition in the case of the Peter-Weyl theorem B.11, with left and right subspaces taken to be $L^2(\mathbb{R})$. Importantly, the involved spaces are now infinite-dimensional and the direct sum is continuous. (5.34) is of course also inspired by the irreducible mass-m representations, indexed by $m \in \mathbb{R}$, which we described earlier.

The decomposition (5.34) in fact even holds representation-theoretically [15]:

Theorem 5.11: Representation Theory of CGal

The left-regular representation $\hat{L}$ of CGal acts on

$$L^2(\mathrm{CGal}) \cong \int_{\mathbb{R}}^{\oplus} \mathrm{d}m \, L^2(\mathbb{R})_m \otimes L^2(\mathbb{R})_m^* \tag{5.35}$$

(where we have added subscript-m labels on the copies of $L^2(\mathbb{R})$) as

$$\hat{L} = \int_{\mathbb{R}}^{\oplus} \mathrm{d}m \, \hat{U}_m \otimes \hat{\mathrm{id}}, \tag{5.36}$$

where $\hat{U}_m$ is the mass-m representation of CGal from definition 5.9. Similarly, the right-regular representation $\hat{R}$ of CGal acts as

$$\hat{R} = \int_{\mathbb{R}}^{\oplus} \mathrm{d}m \, \hat{\mathrm{id}} \otimes \hat{U}_m^*. \tag{5.37}$$

If $\{|m,i\rangle\}_i$ is an orthonormal basis of $L^2(\mathbb{R})_m$ and

$$D_{ij}^{(m)}(g) := \langle m,j| \, \hat{U}_m(g) \, |m,i\rangle \tag{5.38}$$

are the matrix elements of the mass-m representation of CGal, then

$$\begin{aligned} |g\rangle &= \int_{\mathbb{R}} \mathrm{d}m \sum_{i,j} \sqrt{|m|} \, D_{ij}^{(m)}(g) \, |m,i,j\rangle \,, \\ |m,i,j\rangle &:= |m,i\rangle_{L^2(\mathbb{R})_m} \, |m,j\rangle_{L^2(\mathbb{R})_m^*} \,. \end{aligned} \tag{5.39}$$

For a detailed proof, see the appendix of [15];[5] a similar result can be found in [39].

[5] It essentially proceeds by defining improper states (5.39) and then shows that they satisfy all properties of δ-distributions, *i.e.* normalization and transformation behaviour under the right-hand sides of (5.36) and (5.37).

Essentially, the mass m now takes on the role of *charge* in the case of compact groups, and the *left/right subspaces* are now infinite-dimensional irreducible representations rather than finite-dimensional ones as in the compact case. Also, the number $|m|$ replaces the factor $\dim \mathcal{H}_q/|G|$ in the compact case.

The decompositions (5.35), (5.36) and (5.37) show that within $\mathcal{H} = L^2(\mathrm{CGal})$ one may find, with infinite multiplicity, states which transform according to the mass-m representation of definition 5.9 for any m when acted on by $\hat{L}$. Since the decompositions are direct sums and not tensor products, these subspaces are not subsystems. Thus, $L^2(\mathrm{CGal})$ cannot immediately be seen as a collection of massive particles with different masses.[6] This is similar to how in a multi-spin system one may find subspaces transforming under different spin representations [30], but these different spin values do not correspond to the spins composing the system. Rather, $L^2(\mathrm{CGal})$ can be interpreted as describing a *pair of particles* with *equal but variable masses*, one particle susceptible only to $\hat{L}$ and the other only to $\hat{R}$.[7]

The Observers's Own Frame as a Particle. In the framework of quantum reference frame transformations considered in the previous chapters (essentially [15]), we required reduced state of an observer's own reference frame (*i.e.* the state they see of their own frame) to be G-invariant. Using theorem 5.11 we can now understand this requirement further.

By removing any degrees of freedom not invariant under $\hat{L}$, we are forced to mixed states, since the decomposition of $L^2(\mathrm{CGal})$ into irreducible representations does not contain the trivial representation. This also shows again that a coherent perspective-neutral approach (see section 3.5), which intrinsically needs a trivial representation subspace, requires a much larger Hilbert space than $L^2(\mathrm{CGal})$.

Removing those degrees of freedom roughly speaking leaves us, with a single variable-mass particle; this particle is what a perfect observer sees as their own reference frame. More precisely, removing those degrees of freedom and then fixing the mass to a specific value leaves us with a particle of that mass. This is related to the *extra particle* of [15].

[6] It is however possible to decompose any collection of massive particles, *i.e.* a tensor product of mass-m representations into a direct sum of mass-m representations [41].

[7] One can "ignore" either of the particles by essentially removing the left or right factors in $L^2(\mathrm{CGal})$, thereby obtaining a new representation space describing a single variable-mass particle; this space is considered in [39].

6. Quantum Particles as Reference Frames for the Galilei Group

Here we finally apply our framework to quantum particles giving rise to reference frames of the Galilei group. In section 6.1 we overcome the problem of the mass-m representations (5.9) being projective and thus not directly suited for a reference frame, by switching to the centrally extended Galilei group; we will also see that quantum particles always yield imperfect frames for Galilei transformations. Section 6.2 then considers the interesting case where a single quantum particle can be used as a perfect reference frame for any *one-parameter subgroup* of the Galilei group, and two together can theoretically yield even a perfect frame for the whole group; we compare this to the case of a compact group, where a system consisting of two irreducible representations does not have such power. We will however also show that these constructions with quantum particles are unphysical, requiring infinite energy. In section 6.3 we heuristically argue that *squeezed coherent states* are a good choice for classical states, by employing weighted badness measures (recall section 2.5) and entanglement entropy (recall section 4.5), as well as the fact that these states minimize the *Heisenberg uncertainty relation*. Finally, we apply our formalism in section 6.4 to take the view of an imperfect reference frame in a squeezed coherent state, clearly illustrating the "fuzzy view" which we encountered in section 4.5.

6.1 Quantum Particle Galilei Frames are Imperfect

We have seen in section 5.2 that the position or momentum states of a quantum particle cannot provide a perfect reference frame for Gal (even when ignoring the requirement of non-projective representations), because their orbit under Gal was not an orthogonal set. We will show here how quantum particles can be used as reference frames for Galilei transformations and that such a reference frame will always be imperfect.

Quantum Particles as Reference Frames for Gal. Let us first sort out the technicality of requiring non-projective representations in definitions 2.9 and 2.17 of perfect and imperfect quantum reference frames. Mass-m representations of Gal are projective (recall proposition 5.4), and hence can neither be used as perfect nor imperfect reference frames for Gal.

We solve this issue by taking instead $G = \mathrm{CGal}$, where the mass-m representations are non-projective (recall proposition 5.10). Now any mass-m representation clearly yields an imperfect reference frame, because states change only by a phase under θ-translations, and classical states are thus not pairwise orthogonal.

One can however also consider whether a given set of classical states for CGal can distinguish between the different Gal-transformations, *i.e.* ignoring the θ-part of the transformation. More precisely, we call a reference frame for CGal a *perfect quantum reference frame for*

Gal, if the classical states satisfy

$$\langle \Gamma_{\theta',a',v'} | \Gamma_{\theta,a,v} \rangle \propto \delta(a-a')\delta(v-v'). \tag{6.1}$$

Otherwise, we call the reference frame an *imperfect quantum reference frame for* Gal. We will show that the mass-m representations of CGal always give rise to imperfect quantum reference frames for Gal.

Quantum Particles Make Imperfect Frames. With the exact notion of how a mass-m particle can be thought of as a reference frame for Gal in place, we can now show:

Proposition 6.1

Let $|\psi\rangle$ be any state of a quantum particle of mass m, and take it as classical reference frame state for CGal. Then the resulting reference frame will be imperfect for Gal, *i.e.* (6.1) is not satisfied.

Proof. For the resulting frame to be perfect, we require

$$\langle \psi | \hat{U}(a,v) | \psi \rangle = 0, \qquad \forall\, (a,v) \neq (0,0). \tag{6.2}$$

This amounts to (6.1), since θ-translations act only as an additional phase. Assume towards contradiction that (6.2) holds. Writing $|\psi\rangle = \int \mathrm{d}p\, \psi(p)\, |p\rangle$ in the momentum representation, using (5.15) from proposition 5.4, and specializing to $t = 0$, to find that for $\theta = 0$ and $a = 0$ we must have

$$0 = \int \mathrm{d}p\, \mathrm{d}p'\, \psi(p)^* \psi(p + vm). \tag{6.3}$$

Since this must be the case for all $v \neq 0$, taking $v \neq 0$ arbitrarily close to 0 shows that the supports of $\psi(p)$ and $\psi(p + mv)$ must be disjoint; this is only possible for all $v \neq 0$ if $\psi(p) \propto \delta(p - p_0)$ for some $p_0 \in \mathbb{R}$. Thus, $|\psi\rangle \propto |p_0\rangle$ must be a momentum eigenstate. But then $\langle \psi | \hat{U}(a,0) | \psi \rangle \neq 0$ for $a \neq 0$, since translations only change momentum eigenstates by a phase, see (5.13). □

6.2 One-Parameter Subgroups; Pairs of Particles as Perfect Frames

While a single quantum particle cannot make a perfect reference frame for Gal, it can still be used as a perfect reference frame for either boosts or translations. We consider these cases more closely here and show that a quantum particle can even make a perfect reference frame for any one-parameter subgroup of Gal.

We then turn to the fact that *two* quantum particles can make a perfect reference frame for Gal. This for instance works if one particle is a perfect frame for boosts while the other is a perfect frame for translations. Most generally, it is possible with the particles being perfect frames for any two one-parameter subgroups. We will however discover that such constructions are unphysical, requiring infinite energy. Finally, we compare the situation to the well-known case of a compact group.

A Perfect Reference Frame for Boosts. Let us consider the example of momentum eigenstates as a perfect reference for boosts:

Example 6.2

Consider a quantum particle of mass m defining a reference frame for CGal with momentum eigenstates as classical reference frame states:

$$|\Gamma_{0,0,0}\rangle := |p_0\rangle\,, \qquad p_0 \in \mathbb{R}, \tag{6.4}$$

$$|\Gamma_{\theta,a,v}\rangle = \hat{U}_m(\theta,a,v)\,|\Gamma_{0,0,0}\rangle = \mathrm{e}^{im\theta - \mathrm{i}a(p_0+mv/2)}\,|p_0+mv\rangle\,. \tag{6.5}$$

We wish to study the effect of jumping into this reference frame on the observed system S (according to proposition 3.17 one may equivalently consider transforming into this frame from another frame, but jumping is simpler).

So in the external view, let $|\psi\rangle = \int_{\mathbb{R}} \psi(p)\,|p\rangle$ be an imperfect reference frame state and $\hat{\varsigma}_S$ a system states. Using the completeness relation (5.32) from proposition 5.10 we see that the embedding requires the infinite normalization

$$r = \frac{2\pi\cdot\delta(m)|_0}{m}\,\langle\Gamma_{0,0,0}|\Gamma_{0,0,0}\rangle\,, \tag{6.6}$$

making $\hat{E}$ a formal embedding.

Take $|\psi\rangle\langle\psi|_{\tilde{A}}\otimes\hat{\varsigma}_S$ as initial state, and using (4.24), jump into the frame of A (we do not need to specify a state on B for this):

$$\hat{\rho}_{S|R} = \langle\psi|\psi\rangle\int_{\mathrm{CGal}} \mathrm{d}\theta\,\mathrm{d}a\,\mathrm{d}v\,|E(\theta,a,v)|^2\,\hat{U}_S(\theta,a,v)\,\hat{\varsigma}_S\,\hat{U}_S^\dagger(\theta,a,b), \tag{6.7}$$

where

$$|E(\theta,a,v)|^2 = \frac{\left|\langle\Gamma_{\theta,a,v}|\psi\rangle\right|^2}{r\,\langle\psi|\psi\rangle} = \frac{\left|\psi(p_0+mv)\right|^2}{r\,\langle\psi|\psi\rangle}. \tag{6.8}$$

We see that boosts are properly selected according to the momentum-space wave function $\psi(p)$ of the imperfect reference frame state $|\psi\rangle$: if $\psi(p)$ is peaked around a momentum $p_R\in\mathbb{R}$, then the mixture of boosts will be peaked around the boost velocity $v = (p_R - p_0)/m$. Thus, p_0 essentially determines which momentum of the reference frame state should result in the identity boost.

Boost are not the only transformations acting on S: a mixture of all translations and θ-translations acts too. These are however completely unselected, and each of them acts equally. This leads to a state which has been completely mixed by translations and θ-translations. While in physical situations θ-translations often act only as phases, thus not actually changing $\hat{\varsigma}_S$, the same is not true for translations. Unless $\hat{\varsigma}_S$ is translation-invariant, the unselected translations acting on S will mix up the state to some degree and destroy information contained in it. This highlights the fact that momentum eigenstates are completely useless at providing a reference frame for translations, and thus jumping into such a frame is bound to cause noise.

Overall the jump is of course still unitary: Some amount of information must have been transferred into the reference frame state itself in case of a non-translation-invariant $\hat{\varsigma}_S$. As we have remarked in theorem 4.11 only a part of that information is accessible, while the rest is inaccessible, being stored in the exclusively perfect degrees of freedom of the frame.

A Perfect Reference Frame for any Given One-Parameter Subgroup. Since θ-translations act almost trivially on single quantum particles, we directly focus on the one-parameter subgroups of Gal and not on those of CGal. The one-parameter subgroups of Gal are easily seen (recall (5.1)) to be the straight lines through the origin of Gal $\cong \mathbb{R}^2$, and can thus be indexed by an angle $\alpha \in \mathbb{R}/\pi\mathbb{Z}$ (note how rotating a line through the origin by π gives again the same line). Explicitly, they are the subgroups

$$G_\alpha := \big\{(s\cos\alpha, s\sin\alpha) \in \mathrm{Gal} \,:\, s \in \mathbb{R}\big\}, \qquad \alpha \in \mathbb{R}/\pi\mathbb{Z} \tag{6.9}$$

with the group action obtained through restriction, and the parameter being s. We consider one such G_α, and denote a general group element in G_α by

$$g_\alpha(s) := (s\cos\alpha, s\sin\alpha) \in G_\alpha. \tag{6.10}$$

The group action of G_α on $L^2(\mathbb{R})$ is then (see (5.9))

$$\hat{U}(g_\alpha(s)) := \exp(-\mathrm{i}s\hat{h}_{m,\alpha}), \qquad \hat{h}_{m,\alpha} := \cos\alpha\,\hat{p} + \sin\alpha\,\hat{k}_m, \tag{6.11}$$

The generators $\hat{h}_{m,\alpha}$ of one-parameter subgroups G_α are unitarily related:

Proposition 6.3: One Parameter Subgroup Generators

The generators $\hat{h}_{m,\alpha}$ are related among each other through the one-parameter family $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$, $\alpha \in \mathbb{R}$, of unitary operators:

$$\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,\hat{h}_{m,\beta}\,\mathrm{e}^{-\mathrm{i}\alpha\hat{N}} = \hat{h}_{m,\beta+\alpha}, \qquad \forall\,\alpha,\beta \in \mathbb{R}, \tag{6.12}$$

where one defines the "*number*" and *ladder operators*

$$\hat{N} := \hat{a}^\dagger\hat{a}, \qquad \hat{a} := \frac{1}{\sqrt{2m}}(-\hat{k}_m + \mathrm{i}\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m}}(-\hat{k}_m - \mathrm{i}\hat{p}). \tag{6.13}$$

This is shown in appendix A.14. Note that for $t = 0$, the number and ladder operators become the usual number and ladder operators known from the quantum harmonic oscillator [30] of frequency 1, in particular then $\hat{a} = (m\hat{x} + \mathrm{i}\hat{p})/\sqrt{2m}$; hence the names given to $\hat{a}$, $\hat{a}^\dagger$ and $\hat{N}$. This analogy with the harmonic oscillator gives us an intuition for why proposition 6.3 is true: *Time evolution* of a quantum harmonic oscillator is up to a phase given by the inverse of the one-parameter family of operators defined in the proposition, with $t = 0$. Now classically, the time evolution of a harmonic oscillator follows a circle (or ellipse depending on the choice of units) around the origin of phase space. Quantum-mechanically, this is still the case in some sense: when applying the time evolution to $\hat{x}$ (or $\hat{p}$), the result is a circle (ellipse) in the plane of operators spanned by $\hat{x}$ and $\hat{p}$. This is essentially the mechanism behind proposition 6.3. We chose the inverse family, since for us the opposite sense of rotation in phase space is more practical.

Using proposition 6.3, we can construct a set of classical states forming a perfect reference frame for any one-parameter subgroup $G_\alpha \subset$ Gal: Begin with the set of position eigenstates, which form a perfect reference frame for translations, *i.e.* for the subgroup G_0 generated by $\hat{h}_{m,0} = \hat{p}$. Apply then $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$ to this set, producing a new set $\{|x_\alpha\rangle\}_{x\in\mathbb{R}} := \{\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,|x\rangle\}_{x\in\mathbb{R}}$. These form a perfect reference frame for G_α, because

$$\begin{aligned}\hat{U}(g_\alpha(s))\,|x_\alpha\rangle = \exp\big(-\mathrm{i}s\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,\hat{p}\,\mathrm{e}^{-\mathrm{i}\alpha\hat{N}}\big)\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,|x\rangle &= \mathrm{e}^{\mathrm{i}\alpha\hat{N}}\exp\big(-\mathrm{i}s\hat{p}\big)\,|x\rangle \\ &= \mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,|x+s\rangle = |(x+s)_\alpha\rangle\,,\end{aligned} \tag{6.14}$$

(recall (5.12)) and $\langle x_\alpha|x'_\alpha\rangle = \langle x|x'\rangle = \delta(x - x')$ thanks to unitarity of $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$. By a similar argument we find that the set $\{|x_\alpha\rangle\}_{x\in\mathbb{R}}$ is useless as a reference frame for the subgroup $G_{\alpha+\pi/2}$.

To get a more general understanding of the reference frame states $\{|x_\alpha\rangle\}_{x\in\mathbb{R}}$ for G_α, we consider the action of $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$ on phase space:

Proposition 6.4

Let $|\psi_0\rangle$ be any particle state with Wigner distribution W_{ψ_0}. Then $|\psi_\alpha\rangle := \mathrm{e}^{\mathrm{i}\alpha\hat{N}}|\psi_0\rangle$ has the Wigner distribution

$$W_{\psi_\alpha}(x,p) = W_{\psi_0}\left(x\cos\alpha + \frac{p}{m}\sin\alpha, p\cos\alpha - mx\sin\alpha\right). \tag{6.15}$$

We prove this in appendix A.15. Essentially, $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$ "flows" the Wigner function anticlockwise in phase space (from positive x to positive p) along concentric ellipses with axes parallel to the x- and p-axes and a relative scaling of m. If we replace the p-axis by a *velocity axis* (*i.e.* scale it by $1/m$), then the ellipses become concentric circles; see figure 6.1. This is unsurprisingly very similar to the effect of time evolution of the quantum harmonic oscillator, except that the flow is reversed there; the quantum harmonic oscillator is remarkable in that its time evolution in phase space matches the classical *Liouville flow* [30, 55]. Thus, we find that the set $\{x_\alpha\}_{x\in\mathbb{R}}$ are "δ-ridges" in position-velocity space which are rotated by α anticlockwise from the velocity axis; this is also shown in figure 6.1. In phase space, they are additionally stretched by m in p-direction and thus meet the p-axis at a different angle than α. We will later see in section 6.4 that these states can be seen as *infinitely squeezed coherent states*.

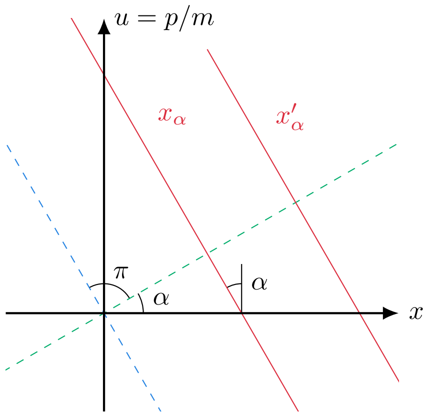


Figure 6.1: The operator $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$ has the effect of rotating a Wigner distribution around the origin of position-velocity space. Shown are the Wigner distributions of two states $|x_\alpha\rangle$ and $|x'_\alpha\rangle$. They are rotated versions of position eigenstates and part of a perfect reference frame for G_α, whose transformations shift position-velocity space parallel to the green line. The Wigner distributions of $|x_\alpha\rangle$ and $|x'_\alpha\rangle$ are completely invariant under transformations in $G_{\alpha+\pi/2}$, which shift position-velocity space parallel to the blue line.

Perfect Gal Frames from Particle Pairs. While one quantum particle cannot make a perfect reference frame for Gal, it can make a perfect frame for any one-parameter subgroup G_α. If this is done, then the particle cannot be used at all to distinguish transformations in the "orthogonal" subgroup $G_{\alpha+\pi/2}$, as its Wigner distribution is invariant under the action of $G_{\alpha+\pi/2}$.

However, adding another particle as a perfect reference frame for $G_{\alpha'}$, where $\alpha' \neq \alpha$, overcomes this problem: together, the particles make a perfect quantum reference frame for Gal.

For example, we may take one particle prepared in a momentum eigenstate as a reference frame for boosts ($G_{\pi/2}$) and the other prepared in a position eigenstate as a reference frame for translations (G_0). Abstractly, this is possible thanks to $L^2(\mathrm{Gal}) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

This approach to perfect reference frames for Gal is not without its problems however: a particle which is a perfect reference frame for a subgroup other than $G_{\pi/2}$ will have a Wigner distribution whose support is unbounded along the p-axis. This means that the kinetic energy $\hat{p}^2/2m$ of such a particle is bound to be infinite, a well-known fact for position eigenstates (a perfect reference frame for G_0). Such two-particle, perfect quantum reference frames for Gal are theoretically admissible, but physically impossible.

Comparison to the Compact Case. Even the very fact that a perfect frame can theoretically be obtained using only two particles is interesting, since for a compact group G, taking the tensor product of two irreducible representations does not yield a perfect frame. In fact, one can show that a sure way of improving a quantum reference frame for G is to add more irreducible representations in order to "fill out" more of $L^2(G)$ [19].

Does "filling out" $L^2(\mathrm{CGal})$ also provide better reference frames? For this we note that considering two particles $L^2(\mathbb{R})\otimes L^2(\mathbb{R})$ as a reference frame for Gal actually already goes a long way in "filling out" $L^2(\mathrm{CGal})$, since our irreducible representations are infinite-dimensional, and $L^2(\mathrm{CGal}) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. The effects of "filling out" $L^2(G)$ are thus not too different between here and the compact case.

What is however different is the size of irreducible representations. In the compact case, they are finite-dimensional (see appendix B.7) and thus taking a pair of *e.g.* spins provides a reference frame of SU(2) which is far from perfect, simply because those spins do not fill out $L^2(\mathrm{SU}(2))$ much. In our case, the irreducible representations are infinite-dimensional and thus two of them happen to already yield very good results, at least in principle.

To summarize, the difference between the compact case and the case of Galilei transformations is the dimension of irreducible representations. The infinite-dimensional irreducible representations of CGal provide much (infinitely) more resources than the finite-dimensional ones in the compact case.

6.3 Choosing Classical States for Single-Particle Frames

Let us return to frames built from a single quantum particle after having briefly considered the intricacies of perfect frames built from two particles. The main goal of this section will be to determine good choices of classical states $|\Gamma_g\rangle$. We will do this heuristically by taking into account weighted badness measures as well as other criteria specifically relevant for quantum particles. This way we will argue that *coherent* or more generally *squeezed coherent states* are good choices for the classical states of imperfect quantum reference frames for the Galilei group.

Weighted Badness Measures. Recall from section 2.5 that the weight $\Omega(g) \geq 0$ of a weighted badness measure must be a class function, that is, it must be constant on conjugacy classes of our group.

Since Gal is Abelian, $\Omega : \mathrm{Gal} \to \mathbb{R}_{\geq 0}$ is simply a function on the group with no further restriction. The centrally extended Galilei group CGal is however not Abelian. To find its conjugacy classes, we compute using (5.27) and (5.29):

$$(\theta', a', v')^{-1} \cdot (\theta, a, v) \cdot (\theta', a', v') = (\theta + a'v - av', a, v). \tag{6.16}$$

By varying a' and v' we see that the conjugacy class of (θ, a, v) is $\{(\varphi, a, v) : \varphi \in \mathbb{R}\}$. This means that for Ω to be a class function, it cannot depend on θ, but may arbitrarily depend on a and v: $\Omega(\theta, a, v) = \Omega(a, v)$.

Furthermore, since θ-translations only provide a phase on states in $\mathcal{H}_{\tilde{R}}$ (see (5.31)), which disappears due to the absolute value taken in the integrand of the badness measure, the integration $\int \mathrm{d}\theta$ yields simply an unimportant multiplicative constant. Any weighted badness measure for CGal-frames therefore essentially reduces to a weighted badness measure for Gal-frames:

$$B(C, \Omega) := \frac{1}{|\langle \Gamma_e | \Gamma_e \rangle|^2} \int_{\mathrm{Gal}} \mathrm{d}g\, \Omega(g) \left| \langle \Gamma_e | \hat{U}_m(g) | \Gamma_e \rangle \right|^2. \tag{6.17}$$

Before arguing for the best classical states, let us see weighted badness measures in action with a simple example:

Example 6.5

Take G_α to be the one-parameter subgroup of Gal, generated by $\cos\alpha\, \hat{p} + \sin\alpha\, \hat{k}_m$ as described in the previous section. If we are only interested in transformations pertaining to G_α, then a good reference frame must only perform well for those transformations. That is, the weight should be of the form

$$\Omega(a, v) = \delta(\cos\alpha\, v - \sin\alpha\, a)\, \tilde{\Omega}(a, v), \tag{6.18}$$

becoming zero outside G_α. For instance, if $\alpha = 0$, G_0 is the subgroup of translations, and $\Omega(a, v) \propto \delta(v)$.

Even more specifically, let us take

$$\Omega(a, v) = \delta(\cos\alpha\, v - \sin\alpha\, a) = \delta(s), \tag{6.19}$$

where s is the parameter of G_α. Then,

$$B(C, \Omega) = \frac{1}{|\langle \Gamma_e | \Gamma_e \rangle|^2} \int \mathrm{d}s \left| \langle \Gamma_e | \hat{U}(g_\alpha(s)) | \Gamma_e \rangle \right|^2. \tag{6.20}$$

The integral will always contain a contribution of 1 at $s = 0$, irrespective of $|\Gamma_e\rangle$. B is thus minimal if the overlap is zero for all $s \neq 0$; we would even have $B = 0$, since the integral is non-zero and also finite only on a set of measure zero. The overlap is zero for $s \neq 0$ if $|\Gamma_e\rangle$ generates a set of pairwise orthogonal classical states under the action of G_α. Minimizing badness with a weight restricting to G_α thus leads us to consider perfect reference frames of G_α, as expected.

We could now choose a specific weight Ω, and solve for the best wave function Γ_e, for instance using variational calculus. Unfortunately, the expression quickly become complicated and issues with integration order arise. Furthermore, choosing a weight is much too specific, since we typically have a rough idea what comprises a "good" or "bad" reference frame in a given situation, but this usually will not be enough to set the precise shape of Ω completely.

It would be better to consider a large family of weights Ω and then to perhaps find suitable seed states more informally and heuristically if needed. Let us do this.

For weights, we will consider at first only rotationally symmetric weights, with a minimum at the identity and monotonously increasing from there. This kind of weight emphasizes large transformations over small ones. This often makes sense physically, since small changes are harder to detect.

Positive Wigner Functions. To assess the badness (6.17) of a reference frame generated from $|\Gamma_e\rangle$ heuristically, it makes sense to consider its Wigner distribution W_{Γ_e}. Let us assume that $\langle\Gamma_e|\Gamma_e\rangle = 1$. Using proposition 5.5 and (5.21), the overlap in the integrand of B is

$$F(a,v) := \left| \langle\Gamma_e| \hat{U}(a,v) |\Gamma_e\rangle \right|^2 = 2\pi \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}p\, W_{\Gamma_e}(x,p) W_{\Gamma_e}(x-a, p-mv). \tag{6.21}$$

Note that this is always positive and

$$\int_{\mathbb{R}} \mathrm{d}a\, \mathrm{d}v\, F(a,v) = \frac{2\pi}{m} \left(\int_{\mathbb{R}} W_{\Gamma_e}(x,p) \right)^2 = \frac{2\pi}{m}. \tag{6.22}$$

In order to minimize (6.17), we thus want $F(a,v)$ to be as strongly peaked around $(a,v) = (0,0)$ as possible. However, Wigner functions cannot be too strongly peaked (in both positive and negative direction) [30], and so the same is true for F, being a kind of *convolution* (more precisely, a *correlation*) of W_{Γ_e} with itself.

The fact that peaks in W_{Γ_e} are not arbitrarily thin combined with the normalization (5.18) of W_{Γ_e} implies that any negative part of W_{Γ_e} "widens" the overall Wigner distribution, since the negative part has to be compensated elsewhere, and neither negative nor positive parts can be too sharp. This further implies that F is widened. If W_{Γ_e} is everywhere non-negative, then no compensation is needed and the Wigner distribution, as well as F can be more strongly peaked.

This heuristic argument leads us to consider only everywhere non-negative Wigner distributions. As we have already mentioned in section 5.2, the pure states corresponding to such distributions are precisely the *squeezed coherent states*; their Wigner distributions are squeezed Gaussians [57].[1] We will describe squeezed coherent states in more detail in the next section.

Our Ω is rotationally symmetric, so we should further only consider rotationally symmetric squeezed coherent states, which are precisely the *coherent* states known as the "most classical states" of the quantum harmonic oscillator [30]. If we allow asymmetric Ω as in figure 2.2, then we consider all squeezed coherent states. But as we will see, squeezed coherent states are described by only few parameters, so solving for the best state is easy in practice.

Differential Entropy. A strongly peaked Wigner distributions W_{Γ_e} is also bound to produce a low differential entropy $H(|E|^2)$ (recall section 4.5) when jumping into a frame which is prepared in a classical state $|\Gamma_g\rangle$: for then $|E(g)|^2 \propto |\langle\Gamma_e|\Gamma_g\rangle|^2 = F(g)$, and strongly peaked probability distributions have low entropy, as they are almost deterministic.

Minimal Uncertainty. Yet another reason to like squeezed coherent states is that they minimize the *Heisenberg uncertainty relation* between the observables corresponding to the major axes of the Wigner distribution in phase space (see next section for details). For a squeezed coherent state $|\psi\rangle$ aligned with $\hat{x}$ and $\hat{p}$ (what we will call *x-p-squeezed coherent*), one for instance has $\langle\hat{\Delta x^2}\rangle_\psi \cdot \langle\hat{\Delta p^2}\rangle_\psi = 1/4$, *i.e.* the uncertainty relation between position and momentum is saturated; we will discuss this further in the next section. Non-squeezed coherent states are rotationally symmetric in phase space and saturate thus all such uncertainty relations. For this reason, non-squeezed coherent states are often thought of as the "most classical states" a particle can be in [30].

[1] We will tke the squeezed coherent states to also include the special case of no squeezing, the *coherent states*. Coherent states have Wigner distributions which are rotationally symmetric Gaussians.

6.4 Quantum Particles in Squeezed Coherent States as Imperfect Frames

We argued in the last section that squeezed coherent states are a good choice for the classical states of an imperfect reference frame for the Galilei group built from a single quantum particle. In this section we will thus first introduce squeezed coherent states in more detail, then take them as classical states, and finally describe the jump into such a reference frame. Some technical details for this section have been moved to appendix B.8.

General Squeezed Coherent States. The non-negative Wigner distributions of pure states are generally rotated and potentially squeezed Gaussian distributions in phase space:

$$W^{\Sigma}_{x_0,p_0}(x,p) = \frac{1}{\pi} \exp\Big[-(x-x_0, p-p_0) \cdot \Sigma \cdot (x-x_0, p-p_0)^T \Big], \tag{6.23}$$

where Σ is a positive semi-definite matrix with $\det \Sigma = 1/4$ [57]. $x_0, p_0 \in \mathbb{R}$ determine the position of the peak of $W^{\Sigma}_{x_0,p_0}$.

The principal axes of Σ dictate the orientation of $W^{\Sigma}_{x_0,p_0}$ in phase space. If they are the x- and p-axis, then we say that $W^{\Sigma}_{x_0,p_0}$ is an *x-p-squeezed coherent state*. It is enough to understand x-p-squeezed coherent states, since all others are simply rotated versions of those (recall the operator $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}$ of proposition 6.4).

***x*-*p*-Squeezed Coherent States.** The x-p-squeezed coherent states have position-space wave functions of the form [30]

$$\psi^{\omega}_{x_0,p_0}(x) := \frac{1}{\sqrt{\omega\sqrt{\pi}}} \exp\left(-\frac{(x-x_0)^2}{2\omega^2} + \mathrm{i}p_0 x \right). \tag{6.24}$$

Here, again $x_0, p_0 \in \mathbb{R}$ labels the position of the peak in phase space, and $\omega > 0$ is the so-called *squeezing parameter*. The corresponding momentum-space wave function is

$$\psi^{\omega}_{x_0,p_0}(p) = \sqrt{\frac{\omega}{\sqrt{\pi}}} \exp\left(-\frac{\omega^2(p-p_0)^2}{2} - \mathrm{i}x_0 p + \mathrm{i}x_0 p_0 \right). \tag{6.25}$$

The meaning of the parameters stems from expectation values:

Proposition 6.6

$$\langle \hat{x} \rangle_{\psi^{\omega}_{x_0,p_0}} = x_0, \qquad \langle \hat{p} \rangle_{\psi^{\omega}_{x_0,p_0}} = p_0, \tag{6.26}$$

$$\langle \hat{\Delta x^2} \rangle_{\psi^{\omega}_{x_0,p_0}} = \frac{\omega^2}{2}, \qquad \langle \hat{\Delta p^2} \rangle_{\psi^{\omega}_{x_0,p_0}} = \frac{1}{2\omega^2}. \tag{6.27}$$

Here,

$$\langle \hat{\Delta A^2} \rangle_{\psi} := \langle (\hat{A} - \langle \hat{A} \rangle_{\psi})^2 \rangle_{\psi} = \langle \hat{A}^2 \rangle_{\psi} - \langle \hat{A} \rangle^2_{\psi} \tag{6.28}$$

is the *variance* of the observable $\hat{A}$ evaluated for the state ψ. The computations needed to arrive at the above proposition are quite straightforward and involve at most a Gaussian integral, so we leave them out.

Thus, ω^2 is the *full width* of the state in position-space, while the full width in momentum-space is $1/\omega^2$. $\omega > 1$ signifies that the state has been *squeezed* to become wider in position-space but thinner in momentum-space; for $\omega < 1$ the opposite occurs; for $\omega = 0$ there is no squeezing and the state is rotationally symmetric in phase space. Note that

$$\langle \hat{\Delta x}^2 \rangle_{\psi^{\omega}_{x_0,p_0}} \cdot \langle \hat{\Delta p}^2 \rangle_{\psi^{\omega}_{x_0,p_0}} = \frac{1}{4} \tag{6.29}$$

saturates the Heisenberg uncertainty relation

$$\langle \hat{\Delta x}^2 \rangle_{\psi} \cdot \langle \hat{\Delta p}^2 \rangle_{\psi} \geq \frac{1}{4}, \tag{6.30}$$

which holds for any state $|\psi\rangle$ of our particle [30]. Conversely, the x-p-squeezed coherent states are the only states with this property; see appendix B.8.

Allowing for non-normalizable states, we find that position and momentum eigenstates can be seen as extreme cases of squeezed coherent states, with $\omega \to 0$ and $\omega \to \infty$ respectively:

$$\psi^{0}_{x_0}(x) \propto \delta(x_0), \tag{6.31}$$
$$\psi^{\infty}_{p_0}(p) \propto \delta(p_0). \tag{6.32}$$

Note that p_0 becomes superfluous in case of a position eigenstate and has hence been omitted as label for the wave function; an intuitive way to see this is that $\langle \hat{\Delta x}^2 \rangle \to 0$ implies $\langle \hat{\Delta p}^2 \rangle \to \infty$, rendering the notion of a momentum expectation value meaningless. For the same reason x_0 becomes superfluous for momentum eigenstates.

Overlaps. For later it will be useful to compute the overlap of two squeezed coherent states:

Proposition 6.7

$$\left\langle \psi^{\omega'}_{x'_0,p'_0} \middle| \psi^{\omega}_{x_0,p_0} \right\rangle = \sqrt{\frac{2\,\omega\omega'}{\omega^2+\omega'^2}} \exp\left[-\frac{(x_0-x'_0)^2 + (\omega\omega')^2(p_0-p'_0)^2}{2(\omega^2+\omega'^2)} + \mathrm{i}\varphi \right], \tag{6.33}$$

$$\varphi = \frac{(p_0-p'_0)\cdot(x_0\omega'^2 + x'_0\omega^2)}{\omega^2+\omega'^2}.$$

This follows from a tedious but straightforward application of Gaussian integrals. Overlaps of squeezed coherent states are exponentially damped with distance between their centres. The combined squeezing $(\omega\omega')^2$ enters by giving more weight to either of the two axes.

Galilei Transformations of Squeezed Coherent States. Squeezed coherent states transform particularly nicely under the mass-m representation of the Galilei group. Namely, they remain squeezed coherent, keep their squeezing ω, but get shifted in phase space as $(x_0, p_0) \mapsto (x_0 + a + vt, p_0 + mv)$, all of which is an immediate consequence of proposition 5.5. In terms of states, we have

Proposition 6.8

$$\hat{U}(a,v) \left|\psi^{\omega}_{x_0,p_0}\right\rangle = \mathrm{e}^{\mathrm{i}\phi} \left|\psi^{\omega}_{x_0+a+vt,p+mv}\right\rangle, \tag{6.34}$$

$$\phi = -(p_0 + mv/2)(a+vt).$$

This follows from (5.14) and (6.24).

Squeezed Coherent Reference Frame states. Let us use squeezed coherent states as classical states in an imperfect reference frame $\tilde{R}$ embedded into a perfect frame R. For concreteness, let us illustrate the "fuzzy view" (recall section 4.5) one sees when jumping or transforming into an imperfect frame. For simplicity, we will take $t = 0$.

So let us assume that

$$|\Gamma_e\rangle := |\psi^{\omega}_{0,0}\rangle \tag{6.35}$$

is some squeezed coherent state with squeezing parameter ω, and which we sensibly centred on the origin of phase space (transforming to a frame in this state then results in a mixture of transformations on S, centred around the identity). Let us further assume that the reference frame is also in a squeezed coherent state, centred on (x, mu) and squeezed by ω':

$$|\psi\rangle_{\tilde{R}} := \left|\psi^{\omega'}_{x,mu}\right\rangle. \tag{6.36}$$

Finally, take any state $\hat{\varsigma}_S$ on S and an unimportant state on B.

Jumping (or transforming) into R and obtaining $\hat{\rho}_{S|R}$, we find a mixture of transformations $\hat{U}^{\dagger}_S(a, v)$ acting on S:

$$\hat{\rho}_{S|R} \propto \frac{\int_{\mathbb{R}} \mathrm{d}\theta}{r} \int_{\mathbb{R}} \mathrm{d}a\, \mathrm{d}v\, |E(a, v)|^2\, \hat{U}^{\dagger}_S(a, v)\, \hat{\varsigma}_S\, \hat{U}_S(a, v), \tag{6.37}$$

where transformations are selected according to

$$\begin{aligned} |E(a, v)|^2 = \frac{1}{r} \left|\left\langle \Gamma_{a,v} \middle| \psi^{\omega'}_{x,mu} \right\rangle\right|^2 &= \frac{1}{r} \left|\left\langle \psi^{\omega}_{a,mv} \middle| \psi^{\omega'}_{x,mu} \right\rangle\right|^2 \\ &= \frac{2\omega\omega'}{r(\omega^2 + \omega'^2)} \exp\left[-\frac{(x-a)^2 + (\omega\omega')^2 m^2 (u-v)^2}{\omega^2 + \omega'^2}\right] \end{aligned} \tag{6.38}$$

(recall section 4.5, and use (6.33) as well as (6.34) from the last two propositions). The mixture of transformations is selected according to a Gaussian: $\hat{\rho}_{S|R}$ contains the original system state $\hat{\varsigma}_S$, Galilei-transformed by (x, u) as most prominent contribution, with other contributions exponentially damped as the transformations depart from (x, u). The relative damping strength along the position and velocity axes is controlled by the combined mass and squeezing term $m^2(\omega\omega')^2$; that the mass has an effect on squeezing is due to a and v having different units. The fact that $|E(a, v)|^2$ has a spread at all illustrates nicely the "fuzziness" of the view in an imperfect reference frame discussed in 4.5. This is shown in figure 6.2.

Note that the integral $\int_{\mathbb{R}} \mathrm{d}\theta$ *cancels* the infinity in r, see (6.6), and the remaining prefactor is finite, up to the normalization of the state on B. Thus, we obtain a physical result despite the infinite factor r required for the centrally extended Galilei group. Also, we could leave out θ as an argument to $|E|^2$, since θ-translations provide only a phase to E and thus do not change $|E|^2$.

The Classical Limit. The Heisenberg uncertainty relation (6.30) dictates the spread of squeezed coherent states; it is the reason why our frames our imperfect. But note how it is an uncertainty relation between position x and *momentum* p, while the task of distinguishing Galilei transformations is concerned with position and *velocity* $u = p/m$. The relevant uncertainty relation is thus

$$\langle \hat{\Delta x^2} \rangle \cdot \langle \hat{\Delta u^2} \rangle \geq \frac{1}{4m^2}. \tag{6.39}$$

This bound can be made arbitrarily small by increasing the mass m. We can see this in action in expression (6.38): If we increase m, the peak in $|E(a, v)|^2$ becomes narrower in the v-direction; we can trade this narrowness with the a-direction by adapting the squeezing ω,

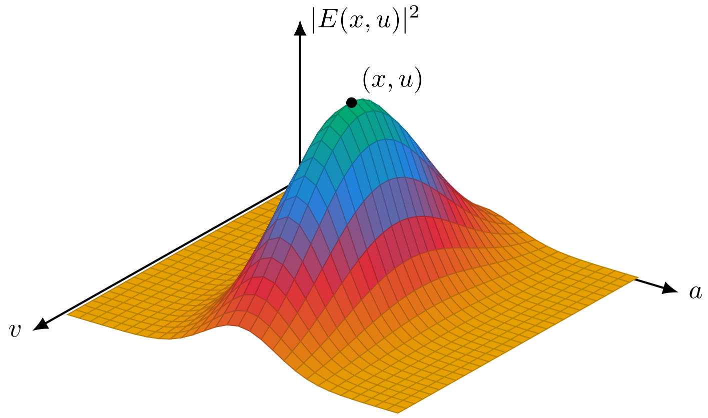


Figure 6.2: Jumping (or transforming) into an imperfect reference frame for the Galilei group. Here, squeezed coherent states with some fixed squeezing have been chosen as classical states, and before the jump the reference frame is prepared in a squeezed coherent state centred on (x,u). After the jump, the system appears transformed by a mixture of Galilei-transformations (a,v), selected according to $|E(a,v)|^2$. $|E(a,v)|^2$ is a squeezed Gaussian centred on (x,u), the Galilei-transformation expected in the case of a perfect frame.

to get an overall narrower peak in $|E(a,v)|^2$. Thus, *heavy quantum particles make for better Galilei reference frames.* In the limit $m \to \infty$, and with arbitrarily strong squeezing our imperfect frames become perfect.

The limit $m \to \infty$ can be seen as a classical limit: think for instance of earth as a reference frame with a huge mass. Classically, we thus expect perfect reference frames for velocity, which makes sense intuitively, since an infinite mass feels no recoil. According to (6.39) we can take $\langle \hat{\Delta x^2} \rangle = \mathcal{O}(1/m)$ and $\langle \hat{\Delta u^2} \rangle = \mathcal{O}(1/m)$ in the limit $m \to \infty$ to get a perfect frame for position and velocity; this introduces an arbitrarily high squeezing, leading to an infinite $\langle \hat{\Delta p^2} \rangle$, and fluctuations in kinetic energy:

$$\langle \hat{\Delta T^2} \rangle = m^2 \langle \hat{u} \rangle^2 \langle \hat{\Delta u^2} \rangle = \mathcal{O}(m). \tag{6.40}$$

Interestingly, the kinetic energy fluctuates as $\mathcal{O}(\sqrt{m})$, which is much smaller than the actual kinetic energy at $\mathcal{O}(m)$. In this sense, the infinite squeezing is even compatible with the above notion of a classical limit.

Note that in a fully classical theory we would expect a perfect frame for any finite m (recall the classical example in section 5.2); there should be no uncertainties at all in position and velocity. This is only possible if the uncertainty relation (6.39) or equivalently (6.30) is completely removed, *i.e.* if $\hbar \to 0$ (which we had set to 1).

Reference Frame in a Cat State. To close this section, let us consider $|\Gamma_e\rangle = \left|\psi^1_{0,0}\right\rangle$ similar to before, but take the *cat state*

$$|\psi\rangle_{\tilde{R}} := \left|\psi^1_{x_1,mu_1}\right\rangle + \left|\psi^1_{x_2,mu_2}\right\rangle \tag{6.41}$$

as a reference frame state. A cat state is a simple model for a mass in spatial superposition, and thus an interesting case for low-energy quantum gravity applications. To prevent unwieldy expressions, we set the squeezing to 1 for all states; this does not change the main features of the result much.

Jumping or transforming into this frame yields

$$
\begin{aligned}
&|E(a,v)|^2 = \\
&\quad \frac{1}{2r\,\langle\psi|\psi\rangle}\Bigg(\exp\left[-\frac{(x_1-a)^2+m^2(u_1-v)^2}{2}\right]+\exp\left[-\frac{(x_2-a)^2+m^2(u_2-v)^2}{2}\right] \\
&\qquad +2\exp\left[-\frac{(x_1-a)^2+m^2(u_1-v)^2+(x_2-a)^2+m^2(u_2-v)^2}{4}\right]\times \\
&\qquad\qquad \times\cos\left[m\frac{(x_1+a)(u_1-v)-(x_2+a)(u_2-v)}{2}\right]\Bigg). \qquad (6.42)
\end{aligned}
$$

Again, this is a tedious but straightforward computation. We observe two distinguished Gaussian peaks around (x_1, u_1) and (x_2, u_2), as well as an oscillating interference term. In a perfect setting, we should only expect two infinitely narrow peaks at (x_1, u_1) and (x_2, u_2). The width of the actually observed peaks is thus a consequence of the imperfection in our frame; it nicely illustrates the "fuzziness" described in section 4.5. The interference term is a consequence of classical reference frame states overlapping and thus also a manifestation of "fuzziness", albeit a less intuitive one. In fact, this term contributes almost nothing to the final expression: If the two peaks are remote, then the interference term is exponentially suppressed, since (a, v) is always far away from at least one of either point (x_1, u_1) or (x_2, u_2); if the two peaks are close, then the cosine is close to one wherever the interference term is not strongly suppressed, and so the interference term merges into the single peak. When the two peaks coincide, then one recovers (6.38) of course. Figure 6.3 shows $|E(a, v)|^2$ for the transformation into a cat state reference frame.

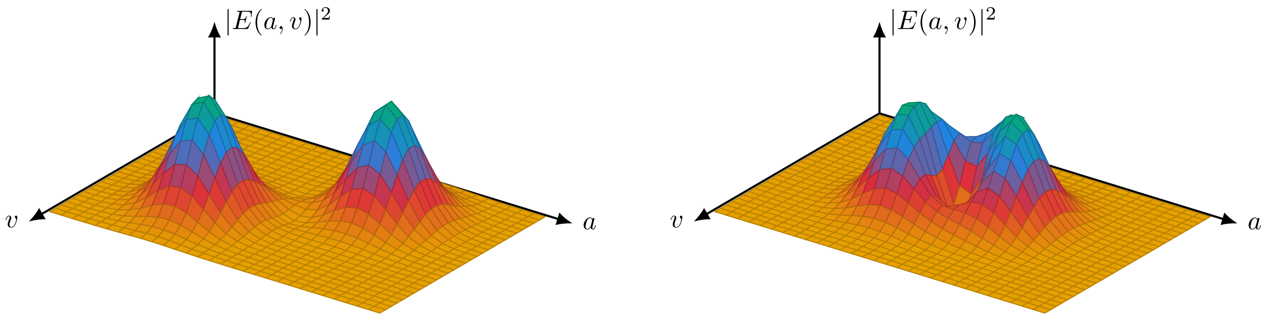


Figure 6.3: $|E(a, v)|^2$ resulting from jumping (or transforming) into a superposition of two squeezed coherent states, with no squeezing and $m = 1$ for simplicity; the classical states are non-squeezed coherent states. Left: If the two states in the cat superposition are not too close, then the result is two roughly Gaussian peaks. Each corresponds to one term in the superposition, and their spread leads to "fuzziness" as in figure 6.2. In a perfect frame, one would expect two perfectly sharp peaks. Right: If the two states in the cat superposition are close to each other, the two peaks begin to merge, and the interference term starts to show.

7. Conclusion and Outlook

We have introduced the framework [15] of quantum reference frame transformations in a new way, from an "observer-first" perspective. Notably, we could derive the existence of the observer-independent external view, only by assuming that the framework should be able to handle the act of "forgetting" an observer. We have worked with the assumption that quantum states containing no absolute degrees of freedom have to commute with the representation of our symmetry group, leading to a non-coherent framework of quantum reference frame transformations. This makes our framework compatible with the rich information theory of quantum reference frames (see *e.g.* [19]). It also distinguishes it from the perspective-neutral approach (*e.g.* [13]), which is in a sense less general since it relies on a coherent notion of invariant states.

We then extended the framework to include the possibility to reversibly transform between imperfect quantum reference frames. This was achieved by embedding imperfect reference frames into perfect ones. We saw that the additional resources of perfect reference frames is really necessary to ensure the unitarity of transformations, even if those resources cannot be accessed by observers. This is not surprising, since Alice might not have access to all information required to reconstruct Bob's point of view, if her reference frame is imperfect. Applying our extended framework, we identify a hallmark of imperfect reference frames: transforming into one produces a "fuzzy view" of physics, with a mixture of transformations, in an appropriate sense centred around the transformations expected from a perfect frame, having acted on the observed system.

Finally, we applied the extended framework to imperfect reference frames of the Galilei group in one dimension, illustrating the "fuzzy view". This provides a possible explanation of what the point of view of a massive quantum particle might look like.

We have thus achieved our goal stated in the introduction: we constructed a framework to handle reversible quantum reference frame transformations between imperfect frames, based on a non-coherent approach. With the ability to model the point of view of massive quantum particles, future applications in low-energy quantum gravity are imaginable: Being able to transform into the frame of a gravitating mass in superposition may help us understand its gravitational field, since in the frame of the mass, one could argue that the field be classical. A very crude thought experiment in this avenue could be a continuation of the idea we showed in figure 5.1:

Consider a Cavendish-type apparatus able to measure the gravitational force exerted by a mass to great precision. Assume that in one frame this mass is in spatial superposition, while the apparatus is localized (or at least, the spatial superposition of the apparatus is insignificant compared to that of the mass). With no theory of quantum gravity we cannot say for sure what the apparatus will read. Transforming into the frame of the mass results in a well-localized mass, and hence a classical gravitational field, but the apparatus is found at least in a mixture of states (in the absence of knowing the exact state structure; also, think of the apparatus as part of the observed system S), producing a mixture of readings. Transforming back to the frame where the apparatus was localized preserves this mixture of readings, if we assume the group (translations in this case) does not have any effect on the readings but only on positions of objects. We have thus obtained a result expected from a superposition of gravitational fields, without assuming the existence of such superpositions.

But of course, there are still open questions:

Firstly, the physical relevance of the state on an observer's own frame is still not clear. We have seen that it surprisingly seems to have an influence on the entanglement structure of states seen by observers: there is a difference between leaving out the observer's own frame when computing reference frame transformations (as in [8]), and keeping it in place, but in a completely mixed state of no information. Further research into the entanglement structure of states after transformations is needed; since this issue does not hinge on the embedding, one may for instance look to the abstract methods employed in [15] for answers.

Secondly, we do not know whether the perfect frames, into which imperfect ones are embedded, are in any sense real, and one may thus similarly wonder about the external view. We may ask: In a universe with only imperfect reference frames, who decides on the state in the external view? And if no one can, can a consensus be reached among the imperfect observers about which states could be the external view state describing the world?

Thirdly, our framework works well with the Galilei group and thus could provide applications to low-energy quantum gravity, essentially because Galilei transformations do not involve transformations of time in any way. In fact, our framework treats time as absolute. Ultimately, this would have to be changed, in order to treat *e.g.* the Poincaré group, but also as a matter of principle to put time and space on equal footing. This of course would touch on the complicated and unsolved issue of time in quantum mechanics.

A. Proofs

We provide here proofs for some statements in the main text.

A.1 Proposition 2.25

Proof of Proposition 2.25. Because G is unimodular, we can substitute $g \rightsquigarrow g'gg'^{-1}$ without changing the Haar measure (see proposition B.3), and thus obtain

$$B(C,\Omega_0) = \frac{1}{|\langle\psi_e|\psi_e\rangle|^2}\int_G \mathrm{d}g\,\mathrm{d}g'\,\Omega_0(g'gg'^{-1})\,\big|\langle\psi_e|\,\hat{U}(g)\,|\psi_e\rangle\big|^2$$
$$= \frac{1}{|\langle\psi_e|\psi_e\rangle|^2}\int_G \mathrm{d}g\,\Omega(g)\,\big|\langle\psi_e|\,\hat{U}(g)\,|\psi_e\rangle\big|^2. \quad \text{(A.1)}$$

Ω is conjugation-invariant, because

$$\Omega(g'^{-1}gg') = \int_G \mathrm{d}g''\,\Omega_0(g''g'^{-1}gg'g''^{-1}) = \int_G \mathrm{d}g''\,\Omega_0(g''gg''^{-1}) = \Omega(g), \quad \text{(A.2)}$$

where we have used the substitution $g'' \rightsquigarrow g''g'$, which leaves the Haar measure invariant. Every class function Ω is also a function on G, and

$$\int \mathrm{d}g'\,\Omega(g'gg'^{-1}) = |G|\cdot\Omega(g), \quad \text{(A.3)}$$

so that we have $B(C,\Omega) \propto B_{\mathrm{conj}}(C,\Omega)$ (up to a possibly infinite constant). □

A.2 Theorem 3.2

Proof of Theorem 3.2. We first argue that in (a) we must have $|\varphi'\rangle_{AB} = |g^{-1}\rangle_A\,|\varphi''\rangle_B$ for some state $|\varphi''\rangle_B$; consequently, also $|\varphi'\rangle_{BA} = |\varphi''\rangle_A\,|g^{-1}\rangle_B$ in equation (3.10). To show this, we begin by noting that generally we can write

$$|\varphi'\rangle_{AB} = \int_G \mathrm{d}g'\,\chi(g')\,|g'\rangle_A\,\big|\varphi'_{g'}\big\rangle_B \quad \text{(A.4)}$$

for non-zero vectors $\big|\varphi'_{g'}\big\rangle_B$ and a function $\chi(g')$. It then follows that

$$\hat{U}^\dagger_{B\to A}\hat{U}^\dagger_{A\to B}\,|\varphi\rangle_A\,|g\rangle_B\,|\psi\rangle_S = \hat{U}^\dagger_{B\to A}\int_G \mathrm{d}g'\,\chi(g')\,|g'\rangle_A\,\big|\varphi'_{g'}\big\rangle_B\,\hat{U}^\dagger(g)\,|\psi\rangle_S$$
$$= \int_G \mathrm{d}g'\,\chi(g')\,\big|\varphi''_{g'}\big\rangle_{AB}\,\hat{U}^\dagger(g'g)\,|\psi\rangle_S \quad \text{(A.5)}$$

for some non-zero states $\left|\varphi''_{g'}\right\rangle_{AB}$. Consider now the case where S carries a regular representation, such that $\hat{U}_S^\dagger(g'g) = \hat{\mathrm{id}}$ only if $g'g = e$. In order to ensure $\hat{U}_{B\to A}\hat{U}_{A\to B} = \hat{\mathrm{id}}$, we must thus have $\chi(g') = \delta(gg')$, *i.e.* $|\varphi'\rangle_{AB} = \left|g^{-1}\right\rangle_A |\varphi''\rangle_B$ with $|\varphi''\rangle_B = |\varphi'_{g^{-1}}\rangle_B$. This must hold also for other systems S, due to (b). The transformation now generally takes the form

$$\hat{U}_{A\to B}^\dagger = \int_G \mathrm{d}g\,\mathrm{d}g'\ |g^{-1}\rangle\langle g'|_A \otimes |\varphi_{g,g'}\rangle\langle g|_B \otimes \hat{U}_S^\dagger(g), \tag{A.6}$$

and $|\varphi_{g,g'}\rangle$ are some unknown vectors. Since $\hat{U}_{A\to B}^\dagger$ is continuous, we must have that $|\varphi_{g,g'}\rangle$ depends continuously on g and g'. Thanks to the principle of relativity, we further have

$$\hat{U}_{B\to A}^\dagger = \int_G \mathrm{d}g\,\mathrm{d}g'\ |\varphi_{g,g'}\rangle\langle g|_A \otimes |g^{-1}\rangle\langle g'|_B \otimes \hat{U}_S^\dagger(g). \tag{A.7}$$

$\hat{U}_{A\to B}^\dagger$ and $\hat{U}_{B\to A}^\dagger$ must both be unitary and inverses of each other. Unitary is equivalent to

$$\hat{U}_{A\to B}\hat{U}_{A\to B}^\dagger = \hat{\mathrm{id}} \tag{A.8}$$

and

$$\hat{U}_{A\to B}^\dagger\hat{U}_{A\to B} = \hat{\mathrm{id}}. \tag{A.9}$$

Because

$$\hat{U}_{A\to B}^\dagger\,|g'\rangle_A\,|g\rangle_B\,|\psi\rangle_S = \left|g^{-1}\right\rangle_A |\varphi_{g,g'}\rangle_B\,\hat{U}_S^\dagger(g)\,|\psi\rangle_S\,, \tag{A.10}$$

the first of these conditions is equivalent to

$$\langle\varphi_{g,g'}|\varphi_{g,g''}\rangle = \delta(g'^{-1}g''), \qquad \forall\, g, g', g'' \in G. \tag{A.11}$$

Thus, $\{|\varphi_{g,g'}\rangle\}_{g'\in G}$ is an orthonormal basis for every $g \in G$, with the same normalization as the basis $\{|g'\rangle\}_{g'\in G}$. We can thus write

$$|\varphi_{g,g'}\rangle = \hat{W}(g)\,|g'\rangle\,, \tag{A.12}$$

where $\hat{W}(g)$ is a family of unitary operators labelled by $g \in G$. To guarantee continuity of $\hat{U}_{A\to B}^\dagger$, $\hat{W}(g)$ must depend continuously on g. This leads us to the form (3.11) of the reference frame transformation. Using (A.6) we write the second condition as

$$\hat{\mathrm{id}} = \int_G \mathrm{d}g\,\mathrm{d}g'\ \left|g^{-1}\right\rangle\left\langle g^{-1}\right|_A \otimes |\varphi_{g,g'}\rangle\langle\varphi_{g,g'}|_B \otimes \hat{\mathrm{id}}_S, \tag{A.13}$$

which upon tracing out A and S is found to be equivalent to

$$\int_G \mathrm{d}g'\ |\varphi_{g,g'}\rangle\langle\varphi_{g,g'}| = \hat{\mathrm{id}}, \qquad \forall\, g \in G. \tag{A.14}$$

Thanks to (A.12) and the completeness relation (2.24), this is guaranteed to hold. Unitarity of $\hat{U}_{A\to B}^\dagger$ is thus equivalent to (A.12). The same is true for the unitarity of $U_{B\to A}^\dagger$.

Finally, we must have that $\hat{U}_{B\to A}^\dagger = \hat{U}_{A\to B}$. Using (A.6) and taking matrix elements, this is equivalent to

$$\langle g''|\varphi_{g,g'}\rangle = \left\langle\varphi_{g^{-1},g''}\middle|g'\right\rangle, \qquad \forall\, g, g', g'' \in G. \tag{A.15}$$

Plugging in (A.12) yields (3.12),

$$\hat{W}^\dagger(g) = \hat{W}(g^{-1}), \qquad \forall\, g \in G. \tag{A.16}$$

□

A.3 Proposition 3.8

Proof of Proposition 3.8. (a) Complete positive follows from

$$ {}_2\langle\psi|\,{}_1\langle\varphi|\,\mathsf{G}_1[\hat\rho]\,|\varphi\rangle_1\,|\psi\rangle_2 = \frac{1}{|G|}\int_G \mathrm{d}g\;{}_2\langle\psi|\,{}_1\langle\varphi|\hat U_1(g)\hat\rho\,\hat U_1^\dagger(g)\,|\varphi\rangle_1\,|\psi\rangle_2 \geq 0 \tag{A.17}$$

for any positive operator $\hat\rho$ on $\mathcal{H}_1\otimes\mathcal{H}_2$, because ${}_2\langle\psi|\,{}_1\langle\varphi|\hat U_1(g)\hat\rho\,\hat U_1^\dagger(g)\,|\varphi\rangle_1\,|\psi\rangle_2 \geq 0$. To show preservation of trace, compute

$$ \mathsf{tr}\big(\mathsf{G}[\hat\rho]\big) = \frac{1}{|G|}\int_G \mathrm{d}g\,\mathsf{tr}\big(\hat U(g)\hat\rho\,\hat U^\dagger(g)\big) = \frac{1}{|G|}\int_G \mathrm{d}g\,\mathsf{tr}(\hat\rho) = \mathsf{tr}(\hat\rho). \tag{A.18}$$

From this it also follows that

$$ \mathsf{tr}_1\big(\mathsf{G}_1[\hat\rho_1\otimes\hat\rho_2]\big) = \mathsf{tr}(\mathsf{G}[\hat\rho_1])\,\hat\rho_2 = \mathsf{tr}(\hat\rho_1)\,\hat\rho_2 = \mathsf{tr}_1(\hat\rho_1\otimes\hat\rho_2), \tag{A.19}$$

which by linearity shows the remark at the end of the proposition.

To show (b), we compute

$$ \mathsf{G}^2\,[\hat\rho] = \frac{1}{|G|^2}\int_G \mathrm{d}g'\,\mathrm{d}g\,\hat U(g'g)\hat\rho\,\hat U^\dagger(g'g) = \frac{|G|}{|G|^2}\int_G \mathrm{d}g\,\hat U(g)\hat\rho\,\hat U^\dagger(g) = \mathsf{G}\,[\hat\rho]\,, \tag{A.20}$$

where we have used the left-invariance (or right-invariance) of the Haar measure.

(c) We have for all $g\in G$ that

$$ \hat U(g)\mathsf{G}[\hat\rho]\hat U^\dagger(g) = \frac{1}{|G|}\int_G \mathrm{d}g'\,\hat U(gg')\hat\rho\,\hat U^\dagger(gg') = \mathsf{G}[\hat\rho], \tag{A.21}$$

again using the invariance of the Haar measure.

(d) If $\hat\rho$ is G-invariant, then it commutes with $\hat U(g)$ for all $g\in G$, and hence $\mathsf{G}[\hat\rho]=\hat\rho$. If $\mathsf{G}[\hat\rho]=\hat\rho$ then according to (c), $\hat\rho$ must be G-invariant.

Finally, it is easily seen that the properties also hold for G_1 in the way described. □

A.4 Proposition 3.9

Proof of Proposition 3.9. Let us compute

$$\begin{aligned} \mathsf{U}^\dagger_{A\to B}\circ\mathsf{G}_A[\,\cdot\,] &= \int_G \mathrm{d}g''\int_G \mathrm{d}g\,\mathrm{d}g'\;|g^{-1}\rangle\langle g''^{-1}g'|_A\otimes|\varphi_{g,g'}\rangle\langle g|_B\otimes\hat U_S^\dagger(g)\,[\,\cdot\,]\times\\ &\qquad\times\int_G \mathrm{d}\bar g\,\mathrm{d}\bar g'\;|g''^{-1}g'\rangle\langle\bar g^{-1}|_A\otimes|\bar g\rangle\langle\varphi_{\bar g,\bar g'}|_B\otimes\hat U_S(\bar g)\\ &= \int_G \mathrm{d}g''\int_G \mathrm{d}g\,\mathrm{d}g'\;|g^{-1}\rangle\langle g'|_A\otimes|\varphi_{g,g''g'}\rangle\langle g|_B\otimes\hat U_S^\dagger(g)\,[\,\cdot\,]\times\\ &\qquad\times\int_G \mathrm{d}\bar g\,\mathrm{d}\bar g'\;|\bar g'\rangle\langle\bar g^{-1}|_A\otimes|\bar g\rangle\langle\varphi_{\bar g,g''\bar g'}|_B\otimes\hat U_S(\bar g). \end{aligned}\tag{A.22}$$

In the second step we have first used the left-invariance of the Haar measure to substitute $g'\rightsquigarrow g''g'$ and $\bar g'\rightsquigarrow g''\bar g'$. On the other hand,

$$\begin{aligned} \mathsf{G}_B\circ\mathsf{U}^\dagger_{A\to B}[\,\cdot\,] &= \int_G \mathrm{d}g''\int_G \mathrm{d}g\,\mathrm{d}g'\;|g^{-1}\rangle\langle g'|_A\otimes\hat L_B(g'')\,|\varphi_{g,g'}\rangle\langle g|_B\otimes\hat U_S^\dagger(g)\,[\,\cdot\,]\times\\ &\qquad\times\int_G \mathrm{d}\bar g\,\mathrm{d}\bar g'\;|\bar g'\rangle\langle\bar g^{-1}|_A\otimes|\bar g\rangle\langle\varphi_{\bar g,\bar g'}|_B\,\hat L_B^\dagger(g'')\otimes\hat U_S(\bar g). \end{aligned}\tag{A.23}$$

These two expressions must be equal. Taking general matrix elements of both expressions shows that their equality is equivalent to

$$\int_G \mathrm{d}g'' \; |\varphi_{g,g''g'}\rangle\langle\varphi_{\bar{g},g''\bar{g}'}| = \int_G \mathrm{d}g'' \, \hat{L}(g'') \, |\varphi_{g,g'}\rangle\langle\varphi_{\bar{g},\bar{g}'}| \, \hat{L}^\dagger(g'') \tag{A.24}$$

for all $g, \bar{g}, g'\bar{g}' \in G$. Substituting $|\varphi_{g,g'}\rangle = \hat{W}(g)\,|g'\rangle$ leads us to

$$\int_G \mathrm{d}g'' \, \hat{W}(g) \, |g''g'\rangle\langle g''\bar{g}'| \, \hat{W}^\dagger(\bar{g}) = \int_G \mathrm{d}g'' \, \hat{L}(g'')\hat{W}(g) \, |g'\rangle\langle\bar{g}'| \, \hat{W}^\dagger(\bar{g})\hat{L}^\dagger(g'') \tag{A.25}$$

for all $g, \bar{g}, g', \bar{g}' \in G$. Since every operator on a reference frame is a linear combination of various $|g'\rangle\langle\bar{g}'|$, this is equivalent to (3.31),

$$\hat{W}(g) \, \mathsf{G}[\,\cdot\,] \, \hat{W}^\dagger(\bar{g}) = \mathsf{G}\big[\hat{W}(g) \, [\,\cdot\,] \, \hat{W}^\dagger(\bar{g})\big], \qquad \forall\, g, \bar{g} \in G. \tag{A.26}$$

□

A.5 Theorem 3.13

Proof of Theorem 3.13. We obtain equation (3.45) by prepending and appending $\mathsf{U}^\dagger_{\to A}$ to both sides of (3.44); (3.44) and (3.45) are thus equivalent. We reproduce the proof in [15] to show (3.45).

Abbreviating $\hat{U}_{AQ} := \big(\hat{L}_A(g) \otimes \hat{U}_Q(g)\big)$, let us compute

$$\begin{aligned} \mathsf{U}^\dagger_{\to A} \circ \mathsf{G}_{RS}[\,\cdot\,] &= \frac{1}{|G|} \int_G \mathrm{d}g \, \hat{U}^\dagger_{\to A} \hat{U}_{AQ}(g) [\,\cdot\,] \, \hat{U}^\dagger_{AQ}(g) \hat{U}_{\to A} \\ &= \frac{1}{|G|} \int_G \mathrm{d}g \, \hat{U}^\dagger_{\to A} \hat{U}_{AQ}(g) \hat{U}_{\to A} \cdot \hat{U}^\dagger_{\to A} [\,\cdot\,] \, \hat{U}_{\to A} \cdot \hat{U}^\dagger_{\to A} \hat{U}^\dagger_{AQ}(g) \hat{U}_{\to A} \end{aligned} \tag{A.27}$$

where in the second step we introduced $\hat{\mathrm{id}} = \hat{U}_{\to A}\hat{U}^\dagger_{\to A}$ on the left and right of $[\,\cdot\,]$. We now observe that

$$\begin{aligned} &\hat{U}^\dagger_{\to A} \hat{U}^\dagger_{AQ}(g) \hat{U}_{\to A} \\ &\quad = \int_G \mathrm{d}g' \, \mathrm{d}g'' \; |g'\rangle\langle g'|_A \, \hat{L}_A(g) \, |g''\rangle\langle g''|_A \otimes \hat{U}_Q(g'^{-1}gg'') \\ &\quad = \int_G \mathrm{d}g' \, \mathrm{d}g'' \; |g'\rangle\langle g'|gg''\rangle\langle g''|_A \otimes \hat{U}_Q(g'^{-1}gg'') \\ &\quad = \int_G \mathrm{d}g'' \; |gg''\rangle\langle g''|_A \otimes \hat{U}_Q((gg'')^{-1}gg'') \\ &\quad = \int_G \mathrm{d}g'' \, \hat{L}_A(g) \, |g''\rangle\langle g''|_A \otimes \hat{\mathrm{id}}_B \otimes \hat{\mathrm{id}}_S \\ &\quad = \hat{L}_A(g) \otimes \hat{\mathrm{id}}_B \otimes \hat{\mathrm{id}}_S. \end{aligned} \tag{A.28}$$

In the last step we have used the completeness relation (2.24). Inserting this observation into (A.27) concludes the proof. □

A.6 Proposition 3.14

Proof of Proposition 3.14. Denote by $\hat{U}_Q$ the representation of G on Q. We compute

$$\begin{aligned}\mathsf{G}_Q\big[\mathsf{tr}_C(\,\cdot\,)\big] &= \frac{1}{|G|}\int_G \mathrm{d}g\,\mathrm{d}g'\,\hat{U}_Q(g)\,\langle g'|_C\,[\,\cdot\,]\,|g'\rangle_C\,\hat{U}_Q^\dagger(g)\\ &= \frac{1}{|G|}\int_G \mathrm{d}g\,\mathrm{d}g'\,\hat{U}_Q(g)\,\langle g'|_C\,\hat{L}_C(g)[\,\cdot\,]\hat{L}_C^\dagger(g)\,|g'\rangle_C\,\hat{U}_Q^\dagger(g)\\ &= \frac{1}{|G|}\int_G \mathrm{d}g\,\mathrm{d}g'\,\,\langle g'|_C\,\big(\hat{U}_Q(g)\otimes\hat{L}_C(g)\big)[\,\cdot\,]\big(\hat{L}_C^\dagger(g)\otimes\hat{U}_Q^\dagger(g)\big)\,|g'\rangle_C\\ &= \mathsf{tr}_C\big(\mathsf{G}_{CQ}[\,\cdot\,]\big).\end{aligned} \tag{A.29}$$

In the second line we have substituted $g' \rightsquigarrow g^{-1}g'$. Also, the resulting state will be G-invariant by the first way of executing F_C. □

A.7 Proposition 3.15

Proof of Proposition 3.15. We have

$$\mathsf{U}_{\to C}[\hat{\rho}_{CQ|C}] = \int_G \mathrm{d}g\,\mathrm{d}g'\,\big(\,|g\rangle\langle g|_C\otimes\hat{U}_Q(g)\big)\hat{\rho}_{CQ|C}\big(\,|g'\rangle\langle g'|_C\otimes\hat{U}_Q^\dagger(g')\big). \tag{A.30}$$

Note that if $\hat{\rho}_{CQ|C}$ is G-invariant on C, *i.e.* $\hat{U}_C(g)\hat{\rho}_{CQ|C}\hat{U}^\dagger(g) = \hat{\rho}_{CQ|C}$ for all $g\in G$, we have

$$\langle g|_C\,\hat{\rho}_{CQ|C}\,|g\rangle_C = \langle e|_C\,\hat{\rho}_{CQ|C}\,|e\rangle_C\,, \qquad \forall\, g\in G. \tag{A.31}$$

Thus,

$$\mathsf{tr}_C\big(\hat{\rho}_{CQ|C}\big) = \int_G \mathrm{d}g\,\,\langle g|_C\,\hat{\rho}_{CQ|C}\,|g\rangle_C = |G|\cdot\langle e|_C\,\hat{\rho}_{CQ|C}\,|e\rangle\,. \tag{A.32}$$

With this in mind, let us compute:

$$\begin{aligned}\mathsf{tr}_C\big(\mathsf{U}_{\to C}[\hat{\rho}_{CQ|C}]\big) &= \int_G \mathrm{d}g\,\big(\,\langle g|_C\otimes\hat{U}_Q(g)\big)\hat{\rho}_{CQ|C}\big(\,|g\rangle_C\otimes\hat{U}_Q^\dagger(g)\big)\\ &= \int_G \mathrm{d}g\,\hat{U}_Q(g)\,\langle e|_C\,\hat{\rho}_{CQ|C}\,|e\rangle_C\,\hat{U}_Q^\dagger(g) = \frac{1}{|G|}\int_G \mathrm{d}g\,\hat{U}_Q(g)\mathsf{tr}_C\big(\hat{\rho}_{CQ|C}\big)\hat{U}_Q^\dagger(g)\\ &= \mathsf{G}_Q\big[\mathsf{tr}_C\big(\hat{\rho}_{CQ|C}\big)\big] = \mathsf{F}_F[\hat{\rho}_{CQ|C}].\end{aligned} \tag{A.33}$$

□

A.8 Theorem 3.16

Proof of Theorem 3.16. Generally, we can write

$$\mathsf{V}^\dagger_{\to A} = \mathsf{U}^\dagger_{\to A}\circ\mathsf{X}, \tag{A.34}$$

where $\mathsf{X}[\,\cdot\,] = \hat{X}[\,\cdot\,]\hat{X}^\dagger$ is a unitarity on AQ. Because $\mathsf{V}^\dagger_{\to A}$ must act equally on all subsystems of Q, and because $\mathsf{U}^\dagger_{\to A}$ does so as well, X must also act on all subsystems of Q in the same way. Now, S is a subsystem of Q, and thus X must act on all subsystems of Q as it acts on S. From theorem 3.2 we know that the transformation $\mathsf{U}^\dagger_{A\to B}$ can only act on

S through the representation $\hat{U}_S$. Any additional action of $\hat{X}$ on S cannot change this. It follows that $\hat{X}$ must be of the form

$$\hat{X} = \int_G \mathrm{d}g\, \hat{X}_A(g) \otimes \hat{U}_Q(g), \tag{A.35}$$

with $\hat{X}_A(g)$ a family of operators on A. Thus, the action of $\hat{U}^\dagger_{A\to B}$ decomposes as follows: $\hat{X}$ applies unitaries $\hat{U}_Q(g)$ on Q, selected by the elements $\hat{X}_A(g)$, then $\hat{U}^\dagger_{\to B}\hat{U}_{\to A}$ is applied, before $\hat{T}_{AB}\hat{X}^\dagger\hat{T}_{AB}$ applies unitaries once more, this time selected according to the state of B. Take for instance the action on $|g_0\rangle_A |e\rangle_B |\psi\rangle_S$ for $g_0 \in G$:

$$\hat{X} |g_0\rangle_A |e\rangle_B |\psi\rangle_S = \int_G \mathrm{d}g\, \hat{X}_A(g) |g_0\rangle_A |g\rangle_B \hat{U}_S(g) |\psi\rangle_S \tag{A.36}$$

and thus

$$U^\dagger_{\to B}\hat{U}_{\to A}\hat{X} |g_0\rangle_A |e\rangle_B |\psi\rangle_S = \int_G \mathrm{d}g\, \left|g^{-1}\right\rangle_A \hat{R}^\dagger_B(g)\hat{X}_B(g) |g_0\rangle_B |\psi\rangle_S \tag{A.37}$$

where we have used the form (3.15); $\hat{X}_B(g)$ is $\hat{X}_A(g)$ acting on B instead of A. Applying $\hat{T}_{AB}\hat{X}^\dagger\hat{T}_{AB}$ completes the action of $\hat{U}^\dagger_{A\to B}$:

$$\hat{U}^\dagger_{A\to B} |g_0\rangle_A |e\rangle_B |\psi\rangle_S = \int_G \mathrm{d}g\, \mathrm{d}g''\, \left|g''^{-1}g^{-1}\right\rangle_A \hat{X}^\dagger_B(g'')\hat{R}^\dagger_B(g)\hat{X}_B(g) |g_0\rangle_B \hat{U}^\dagger_S(g'') |\psi\rangle_S\,. \tag{A.38}$$

We can also compute this directly using theorem 3.2, yielding

$$\hat{U}^\dagger_{A\to B} |g_0\rangle_A |e\rangle_B |\psi\rangle_S = |e\rangle_A \hat{W}(e) |g_0\rangle_B |\psi\rangle_S\,. \tag{A.39}$$

Comparing (A.38) and (A.12) and remembering that $g_0 \in G$ was arbitrary shows that we must have

$$\hat{X}^\dagger_B(g'')\hat{R}^\dagger_B(g)\hat{X}_B(g) = 0 \tag{A.40}$$

if $g'' \neq e$ or $g \neq e$. Let us assume that $\hat{X}_B(g) \neq 0$ for some $g \neq e$. But since $\hat{R}^\dagger_B(g)$ is unitary and since the supports of all $\hat{X}^\dagger_B(g'')$ together must be $\mathcal{H}_B$ (otherwise $\hat{X}$ would not be unitary), we see that (A.40) could then not be satisfied. Hence,

$$\hat{X}_B(g) = 0, \qquad \forall\, e \neq g \in G. \tag{A.41}$$

But $\hat{X}_B(g)$ was just $\hat{X}_A(g)$ acting on B instead of A, so we must also have that $\hat{X}_A(g) = 0$ for all $e \neq g \in G$. Thus, in (A.35) only the term $g = e$ survives, and

$$\hat{X} = \hat{X}_A \otimes \hat{\mathrm{id}}_Q, \tag{A.42}$$

where $\hat{X}_A$ is a unitary on A. More precisely, we must have $\hat{X}_A(g) = \delta(g)\hat{X}_A$ in (A.35). This shows the first statement in the theorem.

From this, compatibility of forgetting observers and jumping follows, because $\hat{X}_A$ is a local unitary and thus does not have any influence after A is traced out. Explicitly:

$$\mathsf{F}_A \circ \mathsf{X}^\dagger = \mathsf{G} \circ \mathsf{tr}_A \circ \mathsf{X}^\dagger = \mathsf{G} \circ \mathsf{tr}_A = \mathsf{F}_A, \tag{A.43}$$

and hence $\mathsf{tr}_C \circ \mathsf{G} \circ \mathsf{U}_{\to A} = \mathsf{tr}_C \circ \mathsf{U}_{\to A} \circ \mathsf{G}_A = \mathsf{F}_A = \mathsf{F}_A \circ \mathsf{X}^\dagger$. If now $\mathsf{V}_{\to A} \circ \mathsf{G}_A = \mathsf{G} \circ \mathsf{V}_{\to A}$, then

$$\mathsf{F}_A = \mathsf{tr}_C \circ \mathsf{G} \circ \mathsf{U}_{\to A} \circ \mathsf{X} = \mathsf{tr}_C \circ \mathsf{G} \circ \mathsf{V}_{\to A} = \mathsf{tr}_C \circ \mathsf{V}_{\to A} \circ \mathsf{G}_A. \tag{A.44}$$

□

A.9 Proposition 3.17

Proof of Proposition 3.17. The remark on G_A is a straightforward consequence of proposition 3.8, with $\mathcal{H}_1 = \mathcal{H}_A$. Also, it is clear from the form requirements of jumps and reference frame transformations that these results must also hold if we switch the roles of Alice and Bob. Let us thus focus on (3.56).

Acting with tr_A on $\mathsf{U}^\dagger_{\to A}[\,\cdot\,]$ (recall (3.18)) gives

$$\mathsf{tr}_A \circ \mathsf{U}^\dagger_{\to A}[\,\cdot\,] = \int_G \mathrm{d}g \left(\langle g|_A \otimes \hat{L}^\dagger_B(g) \otimes \hat{U}^\dagger_S(g) \right) [\,\cdot\,] \left(|g\rangle_A \otimes \hat{L}_B(g) \otimes \hat{U}_S(g) \right). \tag{A.45}$$

Also applying tr_B is easy, because

$$\mathsf{tr}_B \big(\hat{L}^\dagger_B(g) \hat{\rho}_{ABS} \hat{L}_B(g) \big) = \mathsf{tr}_B \big(\hat{\rho}_{ABS} \big), \qquad \forall\, g \in G. \tag{A.46}$$

Thus, we arrive at the right-hand side of (3.56).

Similarly, we find (recall (3.19))

$$\mathsf{tr}_B \circ \mathsf{U}^\dagger_{B\to A}[\,\cdot\,] = \int_G \mathrm{d}g \left(\langle g|_A \otimes \hat{T}_A \hat{R}^\dagger_B(g) \otimes \hat{U}^\dagger_S(g) \right) [\,\cdot\,] \left(|g\rangle_A \otimes \hat{R}_B(g) \hat{T}^\dagger_A \otimes \hat{U}_S(g) \right), \tag{A.47}$$

where $\hat{T}_A$ maps B to A. Applying tr_A then results again in the right-hand side of (3.56). □

A.10 Proposition 4.2

Proof of Proposition 4.2. Using the fact that any vector in $\mathcal{H}_R$ can be expanded in terms of δ-distributions, a general homomorphism $\hat{E} : \mathcal{H}_{\tilde{R}} \to \mathcal{H}_R$ is of the form

$$\hat{E} = \int_G \mathrm{d}g\, |g\rangle\langle \alpha_g|\,, \tag{A.48}$$

where $|\alpha_g\rangle$ are not necessarily normalized vectors. For this to work we must allow for the possibility of $|\alpha_e\rangle$ being non-normalizable too, thus $|\alpha_e\rangle \in \bar{\mathcal{H}}_{\tilde{R}}$; essentially, linearity of $\hat{E}$ dictates only that $\langle \alpha_g|$ acting on the input to $\hat{E}$ must be linear, prompting us to include distributions as candidates for $|\alpha_e\rangle$.

The condition (4.7) contracted with $\langle g'|$ on the left gives $\langle \alpha_{g'}|\, \hat{U}_{\tilde{R}}(g) = \big\langle \alpha_{g^{-1}g'} \big|$, that is

$$|\alpha_{g'g}\rangle = \hat{U}^\dagger_{\tilde{R}}(g'^{-1})\, |\alpha_g\rangle = \hat{U}_{\tilde{R}}(g')\, |\alpha_g\rangle\,. \tag{A.49}$$

Hence,

$$|\alpha_g\rangle = \hat{U}_{\tilde{R}}(g)\, |\alpha_e\rangle\,, \tag{A.50}$$

and the form (4.8) follows.

It is a well-known fact from linear algebra that $\hat{A}^\dagger \hat{A} = \hat{\mathrm{id}}$ for any linear map $\hat{A}$ if and only if $\hat{A}$ is an isometry. Thus, (4.9) is indeed equivalent to $\hat{E}(\alpha_e)$ being an isometry. To show (4.10), we compute

$$\begin{aligned} \hat{E}^\dagger(\alpha_e)\hat{E}(\alpha_e) = \int_G \mathrm{d}g'\, \mathrm{d}g\, |\alpha_g\rangle\langle g'|g\rangle\, \langle \alpha_g| &= \int_G \mathrm{d}g\, |\alpha_g\rangle\langle \alpha_g| \\ &= \int_G \mathrm{d}g\, \hat{U}_{\tilde{R}}(g)\, |\alpha_e\rangle\langle \alpha_e|\, \hat{U}^\dagger_{\tilde{R}}(g), \end{aligned} \tag{A.51}$$

where we have used the orthogonality of perfect reference frame states (2.23). □

A.11 Proposition 4.7

Proof of Proposition 4.7. Consider first the case where $\hat{E}$ is non-formal. $\hat{E}\hat{E}^\dagger$ is an orthogonal projector, because it is Hermitian, and $\hat{E}^\dagger\hat{E} = \hat{\mathrm{id}}_{\tilde{R}}$ shows

$$(\hat{E}\hat{E}^\dagger)^2 = \hat{E}\hat{E}^\dagger\hat{E}\hat{E}^\dagger = \hat{E}\hat{E}^\dagger. \tag{A.52}$$

For $|\psi\rangle_{\tilde{R}} \in \mathcal{H}_{\tilde{R}}$ it holds that

$$\hat{E}\hat{E}^\dagger\hat{E}\,|\psi\rangle_{\tilde{R}} = \hat{E}\,|\psi\rangle_{\tilde{R}}\,, \tag{A.53}$$

hence $\hat{E}\hat{E}^\dagger$ acts as the identity when restricted to $\mathrm{im}(\hat{E})$. Let $|\varphi\rangle_R \in \mathrm{im}(\hat{E})^\perp$ lie in the orthogonal complement of $\mathrm{im}(\hat{E})$, and let $|\chi\rangle_R \in \mathcal{H}_R$. We then find

$$\langle\chi|_R\,\hat{E}\hat{E}^\dagger\,|\varphi\rangle_R = \big(\hat{E}\hat{E}^\dagger\,|\chi\rangle_R\,, |\varphi\rangle_R\big) = 0, \tag{A.54}$$

with $(\,\cdot\,,\,\cdot\,)$ the scalar product on $\mathcal{H}_R$, because $\hat{E}\hat{E}^\dagger\,|\chi\rangle_R \in \mathrm{im}(\hat{E})$. Thus, $\hat{E}\hat{E}^\dagger$ projects onto $\mathrm{im}(\hat{E})$. Consequently, a state $\hat{\rho}_R$ of the perfect frame is an embedded imperfect frame if and only if

$$\mathsf{E}\circ\mathsf{E}^\dagger\,[\hat{\rho}_R] = \hat{\rho}_R. \tag{A.55}$$

Turn now to the case where $\hat{E}$ is formal. Thanks to (4.13) it still holds that $\hat{E}\hat{E}^\dagger\hat{E} = \hat{E}$, and so $\hat{E}\hat{E}^\dagger$ acts as the identity on the image of $\hat{E}$. From this also follows that if $\mathsf{E}\circ\mathsf{E}^\dagger$ does not act as the identity on $\hat{\rho}_R$, then $\hat{\rho}_R$ cannot be an embedded imperfect state. $\square$

A.12 Proposition 5.4

Proof of Proposition 5.4. (a) From the canonical commutation relation (5.4) we straightforwardly find $[\hat{p}, \hat{k}_m] = \mathrm{i}m \cdot \hat{\mathrm{id}}$.

(b) Clearly, $\hat{U}_m$ is unitary as an exponential of i times a Hermitian operator; it remains to show that it is projective. Recall the simple case of the *Baker-Campbell-Hausdorff identity* (see *e.g.* [60]): for operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ it holds that

$$\mathrm{e}^{\hat{A}+\hat{B}}\mathrm{e}^{[\hat{A},\hat{B}]/2} = \mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}. \tag{A.56}$$

We apply it to $\hat{A} = -\mathrm{i}(a'\hat{p} + v'\hat{k}_m)$, $\hat{B} = -\mathrm{i}(a\hat{p} + v\hat{k}_m)$, and thus, using (a),

$$\frac{[\hat{A}, \hat{B}]}{2} = -\frac{\mathrm{i}ma'v}{2} + \frac{\mathrm{i}mav'}{2}, \tag{A.57}$$

to obtain

$$\begin{aligned}\hat{U}_m(a'+a, v'+v)\exp\left(\mathrm{i}\frac{m}{2}(av'-a'v)\right)& \\ = \exp\left(-\mathrm{i}\big[(a+a')\hat{p} + (v+v')\hat{k}_m\big]\right)&\exp\left(\mathrm{i}\frac{m}{2}(av'-a'v)\right) \\ = \exp\left(-\mathrm{i}\big[a'\hat{p} + v'\hat{k}_m\big]\right)\exp\left(-\mathrm{i}\big[a\hat{p} + v\hat{k}_m\big]\right) &= \hat{U}_m(a', v')\hat{U}_m(a, v).\end{aligned} \tag{A.58}$$

(c) Denote by $\hat{U}_m$ and $\hat{U}_{m'}$ the mass-m and mass-m' representation. Assume towards contradiction that there exists a unitary operator $\hat{V}$ such that for every $g \in G$

$$\hat{V}\hat{U}_m(g)\hat{V}^\dagger = \hat{U}_{m'}(g). \tag{A.59}$$

This must in particular hold for the one-parameter subgroups $\hat{U}_m(a,0) = \exp(-\mathrm{i}a\hat{p})$ and $\hat{U}_m(0,v) = \exp(-\mathrm{i}v\hat{k}_m)$, and their m'-conterparts respectively. Taking the derivatives $\mathrm{d}/\mathrm{d}a|_{a=0}$ and $\mathrm{d}/\mathrm{d}v|_{v=0}$ respectively yields

$$\hat{V}\hat{p}\hat{V}^\dagger = \hat{p} \tag{A.60}$$

and

$$\hat{V}\hat{k}_m\hat{V}^\dagger = \hat{k}_{m'}. \tag{A.61}$$

These imply

$$im \cdot \hat{\mathrm{id}} = \hat{V}(\mathrm{i}m \cdot \hat{\mathrm{id}})\hat{V}^\dagger = \hat{V}[\hat{p}, \hat{k}_m]\hat{V}^\dagger = [\hat{p}, \hat{k}_{m'}] = im' \cdot \hat{\mathrm{id}}, \tag{A.62}$$

which in turn implies $m = m'$, a contradiction. Such a unitary $\hat{V}$ cannot exist, and the two representations are inequivalent.

(d) We first compute the actions of pure translations and boosts on position and momentum states, and then use (b) to derive actions of general Galilei transformations. The simplest of the four cases is

$$\hat{U}_m(a,0)\,|p\rangle = \mathrm{e}^{-\mathrm{i}a\hat{p}}\,|p\rangle = \mathrm{e}^{-\mathrm{i}ap}\,|p\rangle\,. \tag{A.63}$$

Using (5.7) we find

$$\langle p|\,\hat{U}_m(a,0)\,|x\rangle = \langle p|\,\mathrm{e}^{-\mathrm{i}a\hat{p}}\,|x\rangle = \mathrm{e}^{-\mathrm{i}ap}\,\langle p|x\rangle = \frac{\mathrm{e}^{-\mathrm{i}(a+x)\,p}}{\sqrt{2\pi}} = \langle p|x+a\rangle\,, \tag{A.64}$$

which through completeness of the improper momentum basis implies

$$\hat{U}_m(a,0)\,|x\rangle = |x+a\rangle\,. \tag{A.65}$$

Because $\hat{k}_m = \hat{p}t - m\hat{x}$ is the sum of two terms whose commutator is proportional to the identity, we can use the formula (A.56) to write

$$\hat{U}_m(0,v) = \mathrm{e}^{-\mathrm{i}vt\hat{p}}\mathrm{e}^{\mathrm{i}mv\hat{x}}\mathrm{e}^{\mathrm{i}mv^2t/2} = \mathrm{e}^{-\mathrm{i}mv\hat{x}}\mathrm{e}^{-\mathrm{i}vt\hat{p}}\mathrm{e}^{-\mathrm{i}mv^2t/2}. \tag{A.66}$$

With this and employing (5.7) we find

$$\langle p|\,\hat{U}_m(0,v)\,|x\rangle = \mathrm{e}^{-\mathrm{i}vtp}\mathrm{e}^{\mathrm{i}mvx}\mathrm{e}^{\mathrm{i}mv^2t/2}\,\langle p|x\rangle = \mathrm{e}^{\mathrm{i}mvx}\mathrm{e}^{\mathrm{i}mv^2t/2}\,\langle p|x+vt\rangle\,, \tag{A.67}$$

and hence

$$\hat{U}_m(0,v)\,|x\rangle = \mathrm{e}^{\mathrm{i}mvx}\mathrm{e}^{\mathrm{i}mv^2t/2}\,|x+vt\rangle\,. \tag{A.68}$$

Similarly,

$$\langle x|\,\hat{U}_m(0,v)\,|p\rangle = \mathrm{e}^{-\mathrm{i}vtp}\mathrm{e}^{\mathrm{i}mvx}\mathrm{e}^{-\mathrm{i}mv^2t/2}\,\langle x|p\rangle = \mathrm{e}^{-\mathrm{i}vtp}\mathrm{e}^{-\mathrm{i}mv^2t/2}\,\langle x|p+mv\rangle\,, \tag{A.69}$$

and thus

$$\hat{U}_m(0,v)\,|p\rangle = \mathrm{e}^{-\mathrm{i}vtp}\mathrm{e}^{-\mathrm{i}mv^2t/2}\,|p+mv\rangle\,. \tag{A.70}$$

Using (b) it is now possible to compute

$$\begin{aligned}\hat{U}_m(a,v)\,|x\rangle = \mathrm{e}^{\mathrm{i}mav/2}\hat{U}_m(a,0)\hat{U}_m(0,v)\,|x\rangle &= \mathrm{e}^{\mathrm{i}mav/2}\mathrm{e}^{\mathrm{i}mvx}\mathrm{e}^{\mathrm{i}mv^2t/2}\,|x+a+vt\rangle\\ &= \mathrm{e}^{\mathrm{i}mv\,(x+a/2+vt/2)}\,|x+a+vt\rangle\,.\end{aligned} \tag{A.71}$$

Similarly, we find

$$\begin{aligned}\hat{U}_m(a,v)\,|p\rangle = \mathrm{e}^{-\mathrm{i}mav/2}\hat{U}_m(0,v)\hat{U}_m(a,0)\,|p\rangle &= \mathrm{e}^{-\mathrm{i}mav/2}\mathrm{e}^{-\mathrm{i}vtp}\mathrm{e}^{-\mathrm{i}mv^2t/2}\mathrm{e}^{-\mathrm{i}ap}\,|p+mv\rangle\\ &= \mathrm{e}^{-\mathrm{i}(a+vt)(p+mv/2)}\,|p+mv\rangle\,.\end{aligned} \tag{A.72}$$

(e) Using (d), we see that $|\psi\rangle = \int_{\mathbb{R}} \psi(x)\,|x\rangle$ is mapped into

$$|\psi'\rangle = \int_{\mathbb{R}} \psi'(x)\,|x\rangle := \hat{U}_m(a,v)\,|\psi\rangle = \int_{\mathbb{R}} \psi(x)\,\mathrm{e}^{\mathrm{i}mv\,(x+a/2+vt/2)}\,|x+a+vt\rangle\,. \tag{A.73}$$

Taking the scalar product with $\langle x|$ we find that the transformed wave function is

$$\psi'(x) = \psi(x-a-vt)\,\mathrm{e}^{\mathrm{i}mv\,(x-a/2-vt/2)}. \tag{A.74}$$

A similar argument works for the momentum-space wave function.

(f) Let us write $|\psi\rangle = \int_{\mathbb{R}} \mathrm{d}p\,\tilde{\psi}(p)\,|p\rangle$ and then compute

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}a\,\mathrm{d}v\,\hat{U}_m(a,v)\,|\psi\rangle\langle\psi|\,\hat{U}_m^{\dagger}(a,v)\\
&\qquad= \int_{\mathbb{R}} \mathrm{d}a\,\mathrm{d}v\,\mathrm{d}p\,\mathrm{d}p'\,\tilde{\psi}(p)\tilde{\psi}(p')^*\,\mathrm{e}^{-\mathrm{i}(a+vt)(p-p')}\,|p+mv\rangle\langle p'+mv|\\
&\qquad= 2\pi\int_{\mathbb{R}} \mathrm{d}v\,\mathrm{d}p\,|\tilde{\psi}(p)|^2\,\,|p+mv\rangle\langle p+mv| = 2\pi\int_{\mathbb{R}} \mathrm{d}p\,\mathrm{d}\bar{p}\,|\tilde{\psi}(p)|^2\,\frac{1}{m}\,|\bar{p}\rangle\langle\bar{p}|\\
&\qquad= \frac{2\pi}{m}\hat{\mathrm{id}}\int_{\mathbb{R}} \mathrm{d}p\,|\tilde{\psi}(p)|^2 = \frac{2\pi}{m}\,\langle\psi|\psi\rangle\cdot\hat{\mathrm{id}}.
\end{aligned} \tag{A.75}$$

In the first equality, we used (5.13), in the second $\int_{\mathbb{R}} \mathrm{d}a\,\exp(-\mathrm{i}aq) = 2\pi\delta(q)$, and in the third we performed the substitution $v \rightsquigarrow \bar{p} = p + mv$.

Employing proposition 2.22, we see that $\hat{U}$ is irreducible. And from proposition 4.4 it then follows that $|\psi\rangle$ can be rescaled into a valid seed state for an embedding. □

To see the significance of the time-dependent boost generator, consider the action of the boost $\hat{U}_m(0,v)$ on the time-dependent momentum state $|p(t)\rangle = \mathrm{e}^{-\mathrm{i}p^2t/2m}\,|p\rangle$:

$$\begin{aligned}
\hat{U}_m(0,v)\,|p(t)\rangle &= \mathrm{e}^{-\mathrm{i}(mv)^2t/2m-\mathrm{i}vpt-\mathrm{i}p^2t/2m}\,\,|p+mv\rangle = \mathrm{e}^{-\mathrm{i}(mv+p)^2t/2m}\,|p+mv\rangle\\
&= |(p+mv)(t)\rangle\,.
\end{aligned} \tag{A.76}$$

So the time-dependence in $\hat{k}_m$ is necessary to provide the correct energy (frequency) to the resulting state, such that the time-independent Schrödinger equation is conserved under the action of the boost. The mass-m representation even conserves the time-dependent Schrödinger equation [30].

A.13 Proposition 5.10

Proof of Proposition 5.10. (a) Clearly, $\hat{U}_m(\theta,a,v)$ is unitary, as $\mathrm{e}^{\mathrm{i}m\theta}$ is a complex phase and $\hat{U}_m(a,v)$ is unitary according to proposition 5.4 (b). We compute

$$\begin{aligned}
\hat{U}_m(\theta',a',v')\,\hat{U}_m(\theta,a,v) &= \exp\left(\mathrm{i}m(\theta'+\theta)\right)\exp\left(\mathrm{i}\frac{m}{2}(av'-a'v)\right)\hat{U}_m(a'+a,v'+v)\\
&= \hat{U}_m\left(\theta'+\theta+\frac{av'-a'v}{2},a'+a,v'+v\right) = \hat{U}_m\big((\theta',a',v')\cdot(\theta,a,v)\big),
\end{aligned} \tag{A.77}$$

and so the representation is non-projective.

(b) If $\hat{U}_m(\theta,a,v)$ were not irreducible, there would exist a true and non-trivial invariant subspace of $L^2(\mathbb{R})$. But this subspace would in particular be invariant under all transformations of the type $\hat{U}_m(0,a,v) = \hat{U}_m(a,v)$. This is impossible, because the mass-m representation of Gal is irreducible according to proposition 5.4 (f). Hence, $\hat{U}_m(\theta,a,v)$ must be irreducible.

(c) Equivalence of $\hat{U}_m(\theta, a, v)$ and $\hat{U}_{m'}(\theta, a, v)$ for $m \neq m'$ would imply equivalence of the mass-m and mass-m' representation of Gal by specifying $\theta = 0$. This is impossible according to 5.4 (c). Thus, $\hat{U}_m(\theta, a, v)$ and $\hat{U}_{m'}(\theta, a, v)$ must be inequivalent.

(d) This differs from the completeness relation in proposition 5.4 (e) only by an additional integral

$$\int_{\mathbb{R}} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}m\theta} \mathrm{e}^{-\mathrm{i}m\theta} = \int_{\mathbb{R}} \mathrm{d}\theta\ = \delta(m)|_0. \tag{A.78}$$

□

A.14 Proposition 6.3

Proof or Proposition 6.3. From (6.13) and (5.10) it follows that

$$[\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{N}, \hat{a}] = -\hat{a}, \qquad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{A.79}$$

The two families $\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\, \hat{a}\, \mathrm{e}^{-\mathrm{i}\alpha\hat{N}}$ and $\mathrm{e}^{-\mathrm{i}\alpha\hat{N}}\, \hat{a}$ of operators, indexed by $\alpha \in \mathbb{R}$, agree for $\alpha = 0$, and their derivatives with respect to α are equal at $\alpha = 0$, thanks to the above commutation relations. Because they also satisfy the *group property*, it follows from proposition A.1 below that the families are equal. Thus, it also holds that

$$\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\, \hat{a}\, \mathrm{e}^{-\mathrm{i}\alpha\hat{N}} = \mathrm{e}^{-\mathrm{i}\alpha}\hat{a}, \tag{A.80}$$

and

$$\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\, \hat{a}^\dagger\, \mathrm{e}^{-\mathrm{i}\alpha\hat{N}} = \mathrm{e}^{\mathrm{i}\alpha}\hat{a}^\dagger. \tag{A.81}$$

With $\hat{p} = -\mathrm{i}\sqrt{m/2}\,(\hat{a} - \hat{a}^\dagger)$ and $\hat{k}_m = -\sqrt{m/2}\,(\hat{a} + \hat{a}^\dagger)$ we find

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\alpha\hat{N}}\, \hat{p}\, \mathrm{e}^{-\mathrm{i}\alpha\hat{N}} &= -\mathrm{i}\sqrt{m/2}\, \left(\mathrm{e}^{-\mathrm{i}\alpha}\hat{a} - \mathrm{e}^{\mathrm{i}\alpha}\hat{a}^\dagger\right) \\
&= \sqrt{m/2}\, \left(-\mathrm{i}\cos(-\alpha)\,\hat{a} + \sin(-\alpha)\,\hat{a} + \mathrm{i}\cos(\alpha)\,\hat{a}^\dagger - \sin(\alpha)\,a^\dagger\right) \\
&\qquad\qquad = \cos\alpha\,\hat{p} + \sin\alpha\,\hat{k}_m = \hat{h}_\alpha.
\end{aligned} \tag{A.82}$$

This implies that $\hat{h}_{m,\alpha} = \cos\alpha\,\hat{p} + \sin\alpha\,\hat{k}_m$ also satisfies the group property:

$$\mathrm{e}^{\mathrm{i}\hat{N}\alpha}\, \hat{h}_{m,\beta}\, \mathrm{e}^{-\mathrm{i}\hat{N}\alpha} = \hat{h}_{m,\beta+\alpha}. \tag{A.83}$$

□

Proposition A.1: One-Parameter Groups of Operators

Let V_α, $\alpha \in \mathbb{R}$, be a family of operators satisfying the *group property*:

$$\hat{V}_\beta \cdot \hat{V}_\alpha = \hat{V}_{\beta+\alpha}, \qquad \forall\, \alpha, \beta \in \mathbb{R}. \tag{A.84}$$

Then $\hat{V}_\alpha$ satisfies the linear, first-order ODE

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{V}_\beta = \hat{V}_\beta \cdot \frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{V}_\alpha\Big|_{\alpha=0} \tag{A.85}$$

and is thus uniquely determined by $\hat{V}_0$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{V}_\alpha\big|_{\alpha=0}$.

Proof. It holds that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{V}_\beta = \frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{V}_{\beta+\alpha}\Big|_{\alpha=0} = \hat{V}_\beta \cdot \frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{V}_\alpha\Big|_{\alpha=0}, \tag{A.86}$$

which is a linear, first-order ODE, if a value for $\frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{V}_\alpha\big|_{\alpha=0}$ is given. By uniqueness of ODE solutions [61], $\hat{V}_\beta$ is then uniquely determined by the initial value $\hat{V}_0$. □

A.15 Proposition 6.4

Proof of Proposition 6.4. To keep the notation simple, we will write ψ_α as ψ, and thus $|\psi\rangle := \mathrm{e}^{\mathrm{i}\alpha\hat{N}}\,|\psi_0\rangle$, with Wigner distribution W_ψ. We will also need

$$\hat{N} = \frac{\hat{p}^2}{2m} + m\hat{x}^2 - \frac{1}{2}, \tag{A.87}$$

which follows from (6.13). The last term only contributes a phase in $|\psi\rangle$, and since Wigner distributions are insensitive to overall phases, we may ignore it. We now compute

$$\begin{aligned}\frac{\partial}{\partial\alpha}W_\psi(x,p) &= \frac{\mathrm{i}m}{4\pi}\int_{\mathbb{R}}\mathrm{d}y\,\Big[(\hat{x}^2\psi)(x-\tfrac{1}{2}y)\psi^*(x+\tfrac{1}{2}y) - \psi(x-\tfrac{1}{2}y)(\hat{x}^2\psi)^*(x+\tfrac{1}{2}y)\Big]\mathrm{e}^{\mathrm{i}py}\\ &+ \frac{\mathrm{i}}{4m\pi}\int_{\mathbb{R}}\mathrm{d}k\,\Big[(\hat{p}^2\tilde{\psi})(p-\tfrac{1}{2}k)\tilde{\psi}^*(p+\tfrac{1}{2}k) - \tilde{\psi}(p-\tfrac{1}{2}k)(\hat{p}^2\tilde{\psi})^*(p+\tfrac{1}{2}k)\Big]\mathrm{e}^{-\mathrm{i}kx}.\end{aligned} \tag{A.88}$$

Here we have used (5.20) in order to split the integration into a part in position-space and a part in momentum space. The position-space part is

$$\begin{aligned}&\frac{\mathrm{i}m}{4\pi}\int_{\mathbb{R}}\mathrm{d}y\,\Big[(x-\tfrac{1}{2}y)^2\psi(x-\tfrac{1}{2}y)\psi^*(x+\tfrac{1}{2}y) - \psi(x-\tfrac{1}{2}y)(x+\tfrac{1}{2}y)^2\psi^*(x+\tfrac{1}{2}y)\Big]\mathrm{e}^{\mathrm{i}py}\\ &\qquad = -\frac{\mathrm{i}mx}{2\pi}\int_{\mathbb{R}}\mathrm{d}y\,y\,\psi(x-\tfrac{1}{2}y)\psi^*(x+\tfrac{1}{2}y)\,\mathrm{e}^{\mathrm{i}py} = -mx\frac{\partial}{\partial p}W_\psi(x,p).\end{aligned} \tag{A.89}$$

Similarly, one finds that the momentum-space part is

$$\frac{p}{m}\frac{\partial}{\partial x}W_\psi(x,p), \tag{A.90}$$

yielding

$$\frac{\partial}{\partial\alpha}W_\psi(x,p) = \frac{p}{m}\frac{\partial}{\partial x}W_\psi(x,p) - mx\frac{\partial}{\partial p}W_\psi(x,p). \tag{A.91}$$

This is a partial differential equation for W_ψ, with α playing the role as time (note also that it strongly resembles the so-called *quantum-Liouville equation* for the harmonic oscillator, which describes time evolution in phase space [30, 55]).

One checks that

$$W_\psi(x,p) := W_{\psi_0}\Big(x\cos\alpha + \frac{p}{m}\sin\alpha, p\cos\alpha - mx\sin\alpha\Big) \tag{A.92}$$

solves the equation:

$$\begin{aligned}\frac{\partial}{\partial\alpha}W_\psi(x,p) &= \partial_1 W_{\psi_0}(x,p)\cdot\Big(-x\sin\alpha + \frac{p}{m}\cos\alpha\Big) + \partial_2 W_{\psi_0}(x,p)\cdot(-p\sin\alpha - mx\cos\alpha)\\ &= -mx\frac{\partial}{\partial p}W_\psi(x,p) + \frac{p}{m}\frac{\partial}{\partial x}W_\psi(x,p).\end{aligned} \tag{A.93}$$

□

B. Details on Various Topics

B.1 Basics of Haar Measures

We give here an overview of Haar measures on a Lie group G, without proofs. As in the main text, let G be a Lie group, *i.e.* a group which is also a *smooth manifold* such that the group multiplication and inverse are smooth maps. In particular this means that G is *locally compact* and *Hausdorff* [33]. If not indicated otherwise, all definitions and statements of this appendix are taken from [34]; one also finds there the proofs which we omitted here.

Definition B.1: Haar Measures and Unimodularity

A *positive regular Borel measure* μ_L on G which is *left-invariant*, *i.e.* for every measurable $E \subset G$ and every $g \in G$ we have

$$\mu_L(gE) = \mu_L(E), \qquad \forall\, E \subset G \text{ measurable}, \quad \forall\, g \in G, \tag{B.1}$$

is called a *left Haar measure*.

If μ_R is a positive regular Borel measure which is *right-invariant*, *i.e.*

$$\mu_R(Eg) = \mu_R(E), \qquad \forall\, E \subset G \text{ measurable}, \quad \forall\, g \in G, \tag{B.2}$$

then μ_R is called a *right Haar measure*.

G is called *unimodular*, if every left Haar measure is also a right Haar measure.

The existence and uniqueness of Haar measures is the content of Haar's theorem:

Theorem B.2: Haar's Theorem

μ_L and μ_R exist and are unique up to positive multiplicative constants.

If G is unimodular, we will always work with $\mu_R = \mu_L$; also, we then have further useful properties:

Proposition B.3: Haar Measure Properties for Unimodular Groups

Let μ be a Haar measure in the case where G is unimodular (thus μ is a left and right Haar measure). μ is then *invariant under inversion*,

$$\mu(E^{-1}) = \mu(E), \qquad \forall\, E \subset G \text{ measurable}, \tag{B.3}$$

as well as *invariant under conjugation*,

$$\mu(g^{-1}Eg) = \mu(E), \qquad \forall\, E \subset G \text{ measurable}, \quad \forall\, g \in G. \tag{B.4}$$

Conjugation invariance is a simple consequence of combined left- and right-invariance, while inversion invariance requires the introduction of the *modular function*.

Finally, it is easy to see that Abelian Lie groups G are unimodular, since left-translation is equal to right-translation by the inverse.

We can define integration with respect to Haar measures ($\int_G \mathrm{d}\mu_L(g)$ or $\int_G \mathrm{d}\mu_R(g)$) in the usual sense of measure theory. If not specified further, we use a left Haar measure and write

$$\mathrm{d}g := \mathrm{d}\mu_L(g). \tag{B.5}$$

Due to proposition B.3 we have in the unimodular case that integrals $\int_G \mathrm{d}g\, f(g)$ of functions f are invariant under the substitutions $g \rightsquigarrow g^{-1}$ and $g \rightsquigarrow g'gg'^{-1}$ for every $g' \in G$. The *total measure* of G is

$$|G| := \mu_L(G) = \int_G \mathrm{d}g, \tag{B.6}$$

and has the following property:

Theorem B.4: Finiteness of Total Measure

$|G| < \infty$ if and only if G is *compact*.

B.2 δ-Distributions on Lie Groups

We turn here to the technical details around δ-distributions on G, in particular needed for constructing the perfect quantum reference frame in theorem 2.16. For this we first generalize the representation (2.11) on $L^2(\mathbb{R})$ to square-integrable functions in $L^2(G, \mu_L)$ and/or $L^2(G, \mu_R)$ (definitions 2.12 and 2.13), and then introduce a notion of δ-distributions on G, analogous to position states $|x\rangle$ in the context of $L^2(\mathbb{R})$.

Intuitively, $-a$ has to be replaced by g^{-1} with $g \in G$; this is not trivial, since G is not necessarily Abelian: both $\psi(g') \mapsto \psi(g^{-1}g')$ and $\psi(g') \mapsto \psi(g'g^{-1})$ for functions $G \ni g' \mapsto \psi(g') \in \mathbb{C}$ and $g \in G$ are *a priori* possibilities. But upon closer inspection only the first option is also a representation, leading us to discard the second. The first option is the so-called *left-regular representation*. If ψ is a function, we denote by $\hat{L}(g)\psi$ the result of acting on it with the left-regular representation of $g \in G$.

If ψ is square-integrable, then so should $\hat{L}(g)\psi$ for all $g \in G$. But since μ_L is only guaranteed to be left-invariant and μ_R only guaranteed to be right-invariant, we can only be sure that $\hat{L}(g)\psi \in L^2(G, \mu_L)$ if $\psi \in L^2(G, \mu_L)$, but *not* that $\hat{L}(g)\psi \in L^2(G, \mu_R)$ if $\psi \in L^2(G, \mu_R)$. This means that the left-regular representation is only well-defined on $L^2(G, \mu_L)$. It is not hard to see that there it conserves scalar products (2.15) and is thus a unitary representation as in definition 2.7.

It is possible to define the so-called *right-regular representation* $\hat{R}(g)$ on $L^2(G, \mu_R)$ as $\psi(g') \mapsto \psi(g'g)$. If $\psi \in L^2(G, \mu_R)$, then $\hat{R}(g)\psi \in L^2(G, \mu_R)$, and $\hat{R}$ is unitary. In example 2.8 the right-regular representation would be $\psi(x) \mapsto \psi(x + a)$.

Overall, this results in definition 2.13.

From definition 2.15 it follows that:

Proposition B.5: δ-Distributions as Generalized Functions

Thinking of δ_c as functions to be integrated against in the sense of (2.20), it holds that

$$\delta_g(g') = \begin{cases} \delta(g^{-1}g') & \text{if } \mu = \mu_L, \\ \delta(g'g^{-1}) & \text{if } \mu = \mu_R. \end{cases} \tag{B.7}$$

$\hat{L}$ and $\hat{R}$ act on δ-distributions as if they were functions, by transforming the arguments as $g' \mapsto g^{-1}g'$ and $g' \mapsto g'g$ respectively. In the same context, we can allow integrations of δ_g's against each other to extend the scalar product (2.15). This results in

$$\langle g'|g\rangle = \langle g|g'\rangle = \begin{cases} \delta(g'^{-1}g) = \delta(g^{-1}g') & \text{if } \mu = \mu_L, \\ \delta(g'g^{-1}) = \delta(gg'^{-1}) & \text{if } \mu = \mu_R. \end{cases} \tag{B.8}$$

It is possible to expand a square-integrable function ψ as

$$|\psi\rangle = \int_G \mathrm{d}\mu(g)\, \psi(g)\, |g\rangle\,, \tag{B.9}$$

and one has the *completeness relation*

$$\int_G \mathrm{d}\mu(g)\, |g\rangle\langle g| = \hat{\mathrm{id}}, \tag{B.10}$$

where $\mathrm{d}\mu(g) = \mathrm{d}\mu_L(g)$ or $\mathrm{d}\mu(g) = \mathrm{d}\mu_R(g)$. If G is unimodular (and $\mu_L = \mu_R$ is chosen), then both notions of δ-distributions coincide.

Proof. Treating integration against δ-distributions semantically as integration of regular functions, we get

$$\int_G \mathrm{d}\mu_L(g')\, \delta_g(g')\psi(g') = \psi(g) = \int_G \mathrm{d}\mu_L(g')\, \delta(g')\psi(gg') = \int_G \mathrm{d}\mu_L(g')\, \delta(g^{-1}g')\psi(g'), \tag{B.11}$$

where we have used left-invariance of μ_L in the last step. Similarly,

$$\int_G \mathrm{d}\mu_R(g')\, \delta_g(g')\psi(g') = \psi(g) = \int_G \mathrm{d}\mu_R(g')\, \delta(g')\psi(g'g) = \int_G \mathrm{d}\mu_R(g')\, \delta(g'g^{-1})\psi(g'), \tag{B.12}$$

using right-invariance of μ_R in the last step. Since these equations hold for every test function ψ, we have shown (B.7).

Furthermore,

$$\langle g'|g\rangle = \int_G \mathrm{d}\mu_L(g'')\delta(g'^{-1}g'')\delta(g^{-1}g'') = \begin{cases} \int_G \mathrm{d}\mu_L(g'')\delta(g'')\delta(g^{-1}g'g'') = \delta(g^{-1}g') \\ \int_G \mathrm{d}\mu_L(g'')\delta(g'^{-1}gg'')\delta(g'') = \delta(g'^{-1}g) \end{cases} \tag{B.13}$$

Here, the upper equality results from substituting $g'' \rightsquigarrow g'g''$, and the lower from $g'' \rightsquigarrow gg''$. Similarly, one shows the case with $\mathrm{d}\mu_R$: the substitutions are now $g'' \rightsquigarrow g''g'$ and $g'' \rightsquigarrow g''g$. We have thus shown (B.8).

Let $|\psi\rangle$ and φ be square-integrable. Compute

$$\begin{aligned} &\left(\int_G \mathrm{d}\mu(g')\, \psi^*(g')\, \langle g'|\right)\left(\int_G \mathrm{d}\mu(g)\, \varphi(g)\, |g\rangle\right) \\ &\quad = \int_G \mathrm{d}\mu(g')\, \mathrm{d}\mu(g)\, \psi^*(g')\varphi(g) \begin{Bmatrix} \delta(g'^{-1}g) \\ \delta(gg'^{-1}) \end{Bmatrix} = \int_G \mathrm{d}\mu(g')\, \mathrm{d}\mu(g)\, \psi^*(g') \begin{Bmatrix} \varphi(g'g) \\ \varphi(gg') \end{Bmatrix} \delta(g) \\ &\quad\quad = \int_G \mathrm{d}\mu(g')\, \psi^*(g')\varphi(g') = \langle\psi|\varphi\rangle\,. \end{aligned} \tag{B.14}$$

Here the upper expressions correspond to $\mu = \mu_L$ and the lower to $\mu = \mu_R$. Similarly, we can show that

$$\langle g'| \int_G \mathrm{d}\mu(g)\, \psi(g)\, |g\rangle = \psi(g'). \tag{B.15}$$

This shows that the decomposition $|\psi\rangle = \int_G \mathrm{d}\mu(g')\psi(g')\,|g\rangle$ is compatible with the bra-ket notation of square-integrable functions.

The completeness relation now follows easily: let $|\psi\rangle$ be a square-integrable function (or even a δ-distribution), and compute

$$\left(\int_G \mathrm{d}\mu(g)\, |g\rangle\langle g|\right) |\psi\rangle = \int_G \mathrm{d}\mu(g)\, |g\rangle\, \psi(g) = |\psi\rangle\,. \tag{B.16}$$

In the unimodular case, both notions of square-integrable functions coincide, and hence also the δ-distributions. Note that as a consequence, $\delta(g^{-1}g') = \delta(g'g^{-1})$ for all $g, g' \in G$; it is also possible to demonstrate this directly using left- and right-invariance of $\mathrm{d}g$. □

B.3 Formal Infinities

We have argued in section 2.4 that certain imperfect quantum reference frames of non-compact groups might need infinite normalization constants. We then outlined how we can introduce such *formally infinite constants* by treating them as *symbols* which are compatible with the arithmetic of complex numbers. In this appendix we will make this precise.

Given an algebraic *field* $\mathbb{F}$ and a *symbol* α, one can consider the field of *rational expressions* in α [62]:

$$\mathbb{F}(\alpha) := \big\{ f(\alpha)/g(\alpha) \;:\; f(\alpha), g(\alpha) \in \mathbb{F}[\alpha],\, g(\alpha) \neq 0 \big\}, \tag{B.17}$$

and $\mathbb{F}[\alpha]$ is the ring of *polynomials* in α with coefficients in $\mathbb{F}$. Multiplication and division, as well as integer powers, are defined via multiplication of polynomials. Together with the addition and subtraction of polynomials, one can define the sum and differences of rational functions. Roughly speaking, this is why $\mathbb{F}(\alpha)$ is again a field. Note that $\mathbb{F}$ is contained in $\mathbb{F}(\alpha)$ as the subset of constant rational expressions, *i.e.* those which do not contain α. It is possible to consider $\mathbb{F}(\alpha, \beta, \gamma, \dots)$ for an arbitrary finite number of symbols.

Given a $\mathbb{F}$-vector space $\mathcal{V}$, we can define a much larger vector space

$$\mathcal{V}_{\alpha,\dots} := \mathrm{span}_{\mathbb{F}(\alpha,\dots)}\, B_{\mathcal{V}}, \tag{B.18}$$

defined as the $\mathbb{F}(\alpha, \dots)$-vector space generated from any basis $B_{\mathcal{V}}$ (*not* a Hilbert basis) of $\mathcal{V}$. Because $\mathbb{F} \subset \mathbb{F}(\alpha, \dots)$, we also have $\mathcal{V} \subset \mathcal{V}_{\alpha,\dots}$. Since any other basis $B'_{\mathcal{V}}$ is a finite linear combination of elements in $B_{\mathcal{V}}$, and since $\mathbb{F}$ is contained in $\mathbb{F}(\alpha, \dots)$, we see that the construction of $\mathcal{V}_{\alpha,\dots}$ is independent of basis chosen.

In our case, $\mathbb{F} = \mathbb{C}$ is the field of complex numbers, and α, β, *etc.* are formal infinities we wish to incorporate into complex number arithmetic. The arithmetic structure inherent in $\mathbb{C}(\alpha)$ provides precisely what we outlined in section 2.4: rational functions in formal infinities are a number system allowing for formal infinities. The vector spaces we are concerned with are typically rigged Hilbert spaces $\mathcal{H} \subset \overline{\mathcal{H}}$, with the scalar product of $\mathcal{H}$ extended to some states of $\overline{\mathcal{H}}$ (for instance, $\mathcal{H} = L^2(G)$, with δ-distributions in $\overline{L^2(G)}$). To complete our treatment of formal infinities, we should really work with the vector space $\overline{\mathcal{H}}_{\alpha,\dots}$ instead of $\overline{\mathcal{H}}$, although we are not as pedantic about this detail in the main text. The scalar product of $\mathcal{H}$, which was already extended to parts of $\overline{\mathcal{H}}$, can now be extended to parts of $\overline{\mathcal{H}}_{\alpha,\dots}$, simply by demanding it to be $\mathbb{C}(\alpha, \dots)$-linear. That is, essentially demanding that one can pull out formal infinities from the scalar product.

B.4 L- and R-Invariant States for Non-Abelian Groups

We saw that the reference frame transformations of example 3.3 do not satisfy (3.30) if G is not Abelian. Technically however, (3.30) must only hold when applied to quantum states, *i.e.* operators which are Hermitian and positive (we allow non-normalizable states, so there is no condition on the trace). We will now take a closer look at this example and show that if G is not Abelian, then there are quantum states for which (3.30) is violated.

Instead of testing (3.31), we now investigate (3.30) directly. One can show that the transformation of example 3.3 satisfies

$$\mathsf{U}^\dagger_{A\to B} \circ \mathsf{G}_A = \mathsf{G}^R_B \circ \mathsf{U}^\dagger_{A\to B}, \tag{B.19}$$

where G^R_B is the G-twirl on B with respect to the right-regular representation $\hat{R}_B$, which, although not corresponding to the chosen canonical representation of G, is also available on $L^2(G)$.[1] To see this, we can retrace the steps in the proof of proposition 3.9 to find that $\mathsf{G}^R_B \circ \mathsf{U}^\dagger_{A\to B}[\,\cdot\,]$ equals (A.23) but with $\hat{L}_B$ replaced by $\hat{R}_B$, and use that $\hat{R}(g'')\,|g^{-1}\rangle = |g'^{-1}g''^{-1}\rangle = |(g''g')^{-1}\rangle$. To summarize: $\mathsf{U}^\dagger_{A\to B}$ maps states which are invariant under $\hat{L}$ (we also say *left-invariant*) to states which are invariant under $\hat{R}$ (*right-invariant*). Prepending and appending $\mathsf{U}^\dagger_{B\to A}$ to (B.19) further yields

$$\mathsf{U}^\dagger_{B\to A} \circ \mathsf{G}^R_B = \mathsf{G}_A \circ \mathsf{U}^\dagger_{B\to A}, \tag{B.20}$$

so the inverse transformation maps right-invariant states to left-invariant states. From this it follows that it is possible to find a left-invariant state $\hat{\rho}'$ such that $\mathsf{U}^\dagger_{A\to B}$ is any right-invariant state $\hat{\rho}$ (simply take $\hat{\rho}' := \mathsf{U}^\dagger_{B\to A}[\hat{\rho}]$). Thus, the reference frame transformations in example 3.3 satisfy (3.30) if and only if every right-invariant state is also left-invariant.

If G is Abelian, then this is the case, because $\hat{R}(g')\,|g\rangle = |gg'^{-1}\rangle = |g'^{-1}g\rangle = \hat{L}^\dagger(g')\,|g\rangle$, and since the Haar measure is inversion-invariant, we have $\mathsf{G} = \mathsf{G}^R$. If the group is non-Abelian, as is for instance the case with the centrally extended Galilei group, then the situation is more complicated, and it is possible to find states which are right-invariant but not left-invariant:

Proposition B.6

Let $g \in G/Z(G)$ be a group element not in the *centre* $Z(G)$ of G. The state

$$\hat{\rho}_{R\not{L}} := \mathsf{G}^R\big[(|e\rangle + |g\rangle)(\langle e| + \langle g|)\big] \tag{B.21}$$

is then right-invariant, but not left-invariant.

Analogously one obtains a left-invariant but not right-invariant state $\hat{\rho}_{L\not{R}}$ by replacing G^R with G.

Proof. Thanks to inversion invariance of the Haar measure, we can write[2]

$$\hat{\rho}_{R\not{L}} = \frac{1}{|G|}\int_G \mathrm{d}g'\,\big(\,|g'\rangle + |gg'\rangle\,\big)\big(\,\langle g'| + \langle gg'|\,\big). \tag{B.22}$$

Acting with $\hat{L}(g'')$ for $g'' \in G$ on this state we obtain

$$\hat{\rho}''_{R\not{L}} := \hat{L}(g'')\hat{\rho}_{R\not{L}}\hat{L}^\dagger(g'') = \frac{1}{|G|}\int_G \mathrm{d}g'\,\big(\,|g''g'\rangle + |g''gg'\rangle\,\big)\big(\,\langle g''g'| + \langle g''gg'|\,\big). \tag{B.23}$$

[1] With the same notation, the G-twirl using the left-regular representation would be $\mathsf{G}^L := \mathsf{G}$, but we will simply write G.

[2] Using inversion invariance, we rewrite $g' \rightsquigarrow g'^{-1}$ for better clarity. The argument however does not depend on this rewriting and works regardless.

We thus have

$$\langle e|\,\hat{\rho}_{R\cancel{L}} = \frac{1}{|G|}\left(2\,\langle e| + \langle g| + \left\langle g^{-1}\right|\right) \tag{B.24}$$

$$\langle e|\,\hat{\rho}''_{R\cancel{L}} = \frac{1}{|G|}\left(2\,\langle e| + \left\langle g''gg''^{-1}\right| + \left\langle g''g^{-1}g''^{-1}\right|\right). \tag{B.25}$$

Since $g \in G/Z(G)$, one can always choose $g'' \in G$ such that $g''gg''^{-1} \neq g$. And since the function $g'' \mapsto g''gg''^{-1}$ is continuous (courtesy of group multiplication and inversion being continuous if G is a Lie group), we can find a g'' such that $g''gg''^{-1} \neq g$ is arbitrarily close to g (in terms of the topology on G), therefore certainly also $g''gg''^{-1} \neq g^{-1}$, and thus $g''g^{-1}g''^{-1} \neq g$.

For such a choice of g'' we obtain

$$\langle e|\,\hat{\rho}_{R\cancel{L}}\,|g\rangle = \frac{\delta(e)}{|G|} \neq 0 = \langle e|\,\hat{\rho}''_{R\cancel{L}}\,|g\rangle\,, \tag{B.26}$$

and thus

$$\hat{\rho}_{R\cancel{L}} \neq \hat{\rho}''_{R\cancel{L}} \tag{B.27}$$

showing that $\hat{\rho}_{R\cancel{L}}$ is not left-invariant. For $\hat{\rho}_{L\cancel{R}}$ the argument is analogous. □

This result shows that for non-Abelian G, example 3.3 cannot be reconciled with the requirement (3.30). Note that the same is true for any family of reference frame transformations satisfying (B.19), *i.e.* all those which map left-invariant states to right-invariant ones, and *vice-versa*. Because of that, we will not pursue such examples further.

Note that for compact groups it is possible to see the existence of states which are right-invariant but not left-invariant directly from the decomposition of the regular representation into irreducible representations of finite dimension, see appendix B.7.

B.5 Traces and Entropies

The completeness relation B.10 allows us to write the partial trace over a Hilbert space of square-integrable functions in a particularly nice way.

For this we first define what we mean by partial trace:

Definition B.7: Partial Trace

Let $\mathcal{H}_X$ and $\mathcal{H}_Y$ be Hilbert spaces, and let $\{|i\rangle\}_i$ be a countable basis of $\mathcal{H}_Y$. Let $\hat{A}_X \otimes \hat{A}_Y : \mathcal{H}_X \otimes \mathcal{H}_Y \to \overline{\mathcal{H}_X \otimes \mathcal{H}_Y}$ be a separable operator (if in any given situation we are not considering rigged Hilbert spaces and the larger spaces do not exist, we simply take $\overline{\mathcal{H}_X \otimes \mathcal{H}_Y} = \mathcal{H}_X \otimes \mathcal{H}_Y$). The *partial trace* of $\hat{A}_X \otimes \hat{A}_Y$ over $\mathcal{H}_X$ is

$$\mathsf{tr}_X\left(\hat{A}_X \otimes \hat{A}_Y\right) := \hat{A}_Y \sum_{i=1}^{\infty} \langle i|\,\hat{A}_X\,|i\rangle\,. \tag{B.28}$$

For a general operator $\hat{A} : \mathcal{H}_X \otimes \mathcal{H}_Y \to \overline{\mathcal{H}_X \otimes \mathcal{H}_Y}$ the partial trace is the linear extension of the above operation, and we may write

$$\mathsf{tr}_X\left(\hat{A}\right) := \sum_{i=1}^{\infty} \langle i|\,\hat{A}\,|i\rangle \tag{B.29}$$

in slight abuse of notation. We allow partial traces to be formally infinite.

An important case, and the main reason for this appendix, will be the trace over Hilbert spaces of square integrable functions on G:

Proposition B.8: Partial Trace over $L^2(G)$

If $\mathcal{H}_X$ is either $L^2(G,\mu_L)$ or $L^2(G,\mu_R)$, then for an operator $\hat{A} : \mathcal{H}_X \otimes \mathcal{H}_Y \to \mathcal{H}_X \otimes \mathcal{H}_Y$,

$$\mathsf{tr}_X(\hat{A}) = \int_G \mathrm{d}\mu(g)\ \langle g|\, \hat{A}\, |g\rangle\,. \tag{B.30}$$

This is similar to how one can take traces in $L^2(\mathbb{R})$ by integrating over the position eigenstate expectation values.

Proof.

$$\sum_i \langle i|\, \hat{A}\, |i\rangle = \int_G \mathrm{d}\mu(g) \sum_i \langle i|g\rangle\langle g|\, \hat{A}\, |i\rangle = \int_G \mathrm{d}\mu(g)\ \langle g|\, \hat{A} \sum_i |i\rangle\langle i|g\rangle = \int_G \mathrm{d}\mu(g)\ \langle g|\, \hat{A}\, |g\rangle\,. \tag{B.31}$$

□

Definition B.7 and proposition B.8 can easily be adapted to the trace instead of the partial trace, by taking $\mathcal{H}_Y = \mathbb{C}$.

An important application of the trace is in the definition of entropy. We adapt the usual definition (see [45]) to accommodate non-normalizable states:

Definition B.9: von Neumann Entropy

For a not necessarily normalized state $\hat{\sigma}$ on a Hilbert space $\mathcal{H}$, *i.e.* $\hat{\sigma}^\dagger = \hat{\sigma}$ and $\hat{\sigma} \geq 0$, we define the (von Neumann) *entropy* as

$$H(\hat{\sigma}) := -\mathsf{tr}\left(\frac{\hat{\sigma}}{\mathsf{tr}\,\hat{\sigma}} \log_2 \frac{\hat{\sigma}}{\mathsf{tr}\,\hat{\sigma}}\right). \tag{B.32}$$

We use the convention that $0 \cdot \log_2 0 = 0$, which is consistent with $x \mapsto x \log_2 x$ being continuously extended for $x \to 0$.

Clearly, this definition matches the usual definition $H(\hat{\sigma}) = -\mathsf{tr}\big(\hat{\sigma} \log_2 \hat{\sigma}\big)$ in the case where the state is normalized to $\mathsf{tr}\,\hat{\sigma} = 1$.

To see that the adapted entropy is indeed useful, let us compute it for a couple of non-normalizable states. Especially useful for us will be linear combinations of δ-distributions on a unimodular Lie group G:

Example B.10

Let G be unimodular.

(a) The pure (improper) state $|g\rangle\langle g|$, $g \in G$, satisfies $\mathsf{tr}(|g\rangle\langle g|) = \delta(e)$. When acting on operators, $\log_2$ is defined by acting on eigenvalues in a diagonal form of its argument. With this we find the entropy to be

$$\begin{aligned} H(|g\rangle\langle g|) = -\int_G \mathrm{d}g'\ \langle g'|\, \frac{|g\rangle\langle g|}{\delta(e)} \log_2 \frac{|g\rangle\langle g|}{\delta(e)}\, |g'\rangle &= \frac{1}{\delta(e)} \log_2(1)\, \langle g|\, \frac{|g\rangle\langle g|}{\delta(e)}\, |g\rangle \\ &= 1 \cdot \log_2(1) = 0. \end{aligned} \tag{B.33}$$

As expected, the entropy of the state is zero.

(b) Take the equally pure state $|\psi\rangle = (|g_1\rangle + |g_2\rangle)/\sqrt{2}$ for $g_1, g_2 \in G$, $g_1 \neq g_2$. The trace of the corresponding density operator $\hat{\sigma}$ is also $\delta(e)$. We have, formally speaking,

$$\log_2 \frac{|\psi\rangle\langle\psi|}{\delta(e)} = \log_2(1)\frac{|\psi\rangle\langle\psi|}{\delta(e)} + \log_2(0)\frac{(|g_1\rangle - |g_2\rangle)(\langle g_1| - \langle g_2|)}{2\delta(e)} + \log_2(0)\cdot\hat{A}, \tag{B.34}$$

where $\hat{A}$ is an unimportant self-adjoint operator which annihilates both $|g_1\rangle$ and $|g_2\rangle$. The integral $\int_G \mathrm{d}g'$ in the expression for the entropy forces $g' = g_1$ or $g' = g_2$, so that we obtain effectively the trace over a two-dimensional subspace:

$$H(\hat{\sigma}) = -\frac{1}{4}\log_2(0)\cdot \mathsf{tr}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{4}\log_2(0)\cdot\mathsf{tr}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0. \tag{B.35}$$

Note that all factors of $\delta(e)$ have cancelled thanks to $\langle g|g'\rangle = \delta(g^{-1}g')$. As expected, the entropy is again zero.

The same holds for other finite superposition of δ-states.

(c) Consider finally the mixed state $\hat{\sigma} = (|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|)/2$ with trace $\delta(e)$. We have

$$\log_2 \frac{\hat{\rho}}{\delta(e)} = \log_2\left(\frac{1}{2}\right)\frac{1}{\delta(e)}(|g_1\rangle\langle g_2| + |g_2\rangle\langle g_2|) + \log_2(0)\cdot\hat{A}, \tag{B.36}$$

where $\hat{A}$ is as in the above example. Thus, again keeping track of factors of $\delta(e)$, we get

$$H(\hat{\sigma}) = -\frac{1}{2}\log_2\left(\frac{1}{2}\right)\cdot\mathsf{tr}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -\log_2\left(\frac{1}{2}\right) = 1. \tag{B.37}$$

As expected the entropy is one bit, corresponding to a "yes/no" probability distribution with equally likely outcomes.

B.6 Haar Measures for the Galilei Groups

Let μ_n be the Lebesgue measure on $\mathbb{R}^n$, *i.e.* the measure obtained through Lebesgue integration. We will show that μ_2 is a left- and right Haar measure for Gal = Gal(1), and that μ_3 is a left- and right Haar measure for CGal = CGal(1). We then fix the Haar measures used in the main text.

As claimed in the main text, Gal is unimodular with Haar measure equal to μ_2:

Proof of Proposition 5.2. Gal is Abelian and hence unimodular. Note then that the left action of $g = (a, v) \in$ Gal on any measurable set $B \subset \mathbb{R}^2 \cong$ Gal is simply a translation by (a, v) in $\mathbb{R}^2$. This evidently leaves $\mu_2(B)$ invariant, showing that μ_2 is a left Haar measure. □

We will always use μ_2 as a Haar measure for Gal and write $\mathrm{d}g = \mathrm{d}\mu_2(g)$. If now $f : \mathrm{Gal} \to \mathbb{C}$ then we can perform the integral

$$\int_{\mathrm{Gal}} \mathrm{d}\mu_2(g)\, f(g) = \int \mathrm{d}a\,\mathrm{d}v\, f(a, v) \tag{B.38}$$

over translations and boosts separately. This particularly implies that

$$\delta\big((a,v)\big) = \delta(a)\delta(v), \tag{B.39}$$

where the δ-distribution on the left-hand side is to be understood in the sense of definition 2.15, and the two δ-distributions on the right-hand side are the usual δ-distributions on $\mathbb{R}$. The total measure of the group is

$$|\mathrm{Gal}| = \int \mathrm{d}a\,\mathrm{d}v = \infty. \tag{B.40}$$

It is infinite, Gal being *non-compact*, in line with theorem B.4.

Similarly, CGal is unimodular with μ_3 as Haar measure:

Proof of Proposition 5.8. The left group action of $g = (\theta, a, v) \in \mathrm{CGal}$ on a measurable subset $B \subset \mathbb{R}^3 \cong \mathrm{CGal}$ induces a translation of B in $\mathbb{R}^3$ by (θ, a, v), and a translation in the θ-direction linearly depending on a and v. The former translation clearly keeps $\mu(B)$ invariant (by the same argument as for Gal). The latter translation introduces a *linear shear* along θ, *i.e.* a rectangular box in θ-a-v space is deformed into a prism. In general, this does not change $\mu(B)$ according to *Cavalieri's principle* (which holds more generally for Lipschitz continuous shears). Thus, μ_3 is a left Haar measure. Similarly, we can show that it is a right Haar measure, making CGal unimodular (note that CGal is not Abelian). ☐

We always use μ_3 as Haar measure for CGal and write $\mathrm{d}g := \mathrm{d}\mu_3(g)$, *i.e.* $\mathrm{d}(\theta, a, v) = \mathrm{d}\theta\,\mathrm{d}a\,\mathrm{d}v$, and $\delta\big((\theta, a, v)\big) = \delta(\theta)\delta(a)\delta(v)$. Accordingly, $|\mathrm{CGal}| = \infty$ (CGal is non-compact). Note however that we will have to distinguish the infinities $|\mathrm{Gal}|$ and $|\mathrm{CGal}|$.

Finally, we note that the Haar measures of Gal and CGal are inversion- and conjugation-invariant, thanks to proposition B.3.

B.7 Representation Theory of Compact Groups

The central result in the representation theory of compact topological groups is the *Peter-Weyl theorem* [41]. A form of the theorem useful for applications in quantum reference frames is:

Theorem B.11: Peter-Weyl: Representation Theory of Compact Groups

(a) One can decompose

$$L^2(G) \cong \bigoplus_q \mathcal{H}_q \otimes \mathcal{H}_q^*, \tag{B.41}$$

where the index q ranges over the equivalence classes of *finite-dimensional*, *irreducible* representations $\hat{U}_q$ of G, $\mathcal{H}_q$ is the representation space of the q-th representation, and $\mathcal{H}_q^*$ is its complex conjugate.

(b) The left-regular representation $\hat{L}$ on $L^2(G)$ acts as the q-th irreducible representation on $\mathcal{H}_q$ and trivially on all $\mathcal{H}_q^*$, *i.e.*

$$\hat{L} = \bigoplus_q \hat{U}_q \otimes \hat{\mathrm{id}}_q. \tag{B.42}$$

(c) The right-regular representation $\hat{R}$ on $L^2(G)$ acts as the *complex conjugate* of the q-th irreducible representation on $\mathcal{H}_q^*$ and trivially on all $\mathcal{H}_q$, *i.e.*

$$\hat{R} = \bigoplus_q \hat{\mathrm{id}}_q \otimes \hat{U}_q^*. \tag{B.43}$$

Accordingly, the $\mathcal{H}_q$ are also called the *left subspaces*, and the $\mathcal{H}_q^*$ the *right subspaces*.

(d) Let $\{|q,x\rangle\}_x$ be an orthonormal basis of $\mathcal{H}_q$, and denote by

$$D_{xy}^{(q)}(g) := \langle q,y| \hat{U}_q(g) |q,x\rangle \tag{B.44}$$

the matrix elements of the irreducible representation labelled by q. With this it is possible to write the improper basis of δ-distributions as

$$|g\rangle = \sum_{q,x,y} \sqrt{\frac{\dim \mathcal{H}_q}{|G|}}\, D_{x,y}^{(q)}(g)\, |q,x,y\rangle\,, \qquad |q,x,y\rangle := |q,x\rangle_{\mathcal{H}_q} |q,y\rangle_{\mathcal{H}_q^*}\,. \tag{B.45}$$

Note that because G is compact, $|G| < \infty$.

The Peter-Weyl theorem is usually stated in a different form, in terms of the matrix elements $D_{xy}^{(q)}(g)$ [41]. This is however equivalent to the decompositions of the Hilbert space in (a) and of the representations in (b) and (c) [44]. (a) - (c) is equivalent to (d), see *e.g.* [23]. See also the applications of the theorem in [13, 15, 19, 20]. In quantum physics, we often call the index q the *charge* of the corresponding irreducible representation. *E.g.* for SU(2), the charge is the *total spin* j. The spaces $\mathcal{H}_i$ and $\mathcal{H}_i^*$ are also called the *colour* and *flavour* subspaces [23].

B.8 Squeezed Coherent States

We provide here some details left out in section 6.4.

Heisenberg Uncertainty. Any state $|\psi\rangle$ of our quantum particle satisfies the *Heisenberg uncertainty relation* (6.30). This is a special case of the more general uncertainty relation [30]

$$\langle \hat{\Delta A}^2\rangle_\psi \langle \hat{\Delta B}^2\rangle_\psi \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle_\psi\right|^2 \tag{B.46}$$

valid for any two observables $\hat{A}$ and $\hat{B}$.

The general Heisenberg uncertainty relation (B.46) is commonly proved by considering the Hermitian operators $\hat{\Delta A} := \hat{A} - \langle\hat{A}\rangle_\psi\,\hat{\mathrm{id}}$, $\hat{\Delta B} := \hat{B} - \langle\hat{B}\rangle_\psi\,\hat{\mathrm{id}}$, and computing

$$\begin{aligned}\left|\langle[\hat{A},\hat{B}]\rangle_\psi\right| &= \left|\langle[\hat{\Delta A},\hat{\Delta B}]\rangle_\psi\right| = \Big|\, \langle\psi|\, \hat{\Delta A}^\dagger \hat{\Delta B}\,|\psi\rangle - \langle\psi|\,\hat{\Delta B}^\dagger\hat{\Delta A}\,|\psi\rangle\,\Big| \\ &\leq 2\Big|\,\langle\psi|\,\hat{\Delta A}^\dagger\hat{\Delta B}\,|\psi\rangle\,\Big| \leq 2\sqrt{\langle\psi|\,\hat{\Delta A}^\dagger\hat{\Delta A}\,|\psi\rangle}\,\sqrt{\langle\psi|\,\hat{\Delta B}^\dagger\hat{\Delta B}\,|\psi\rangle}. \end{aligned} \tag{B.47}$$

In the first step we have used that $\hat{\mathrm{id}}$ commutes with every other operator, and in the second step, that $\hat{\Delta A}$ and $\hat{\Delta B}$ are Hermitian. The first inequality is the *triangle inequality*, in the form $|z - z^*| \leq 2|z|$ for any $z \in \mathbb{C}$, here with $z = \langle\psi|\,\hat{\Delta A}^\dagger\hat{\Delta B}\,|\psi\rangle$. The second inequality is the *Cauchy-Schwarz inequality* in the form $|\,\langle\phi|\phi'\rangle\,| \leq \sqrt{\langle\phi|\phi\rangle}\sqrt{\langle\phi'|\phi'\rangle}$ for any two states $|\phi\rangle$ and $|\phi'\rangle$, here with $|\phi\rangle = \hat{\Delta A}\,|\psi\rangle$ and $|\phi'\rangle = \hat{\Delta B}\,|\psi\rangle$. The inequality (B.46) follows by squaring both sides of (B.47) and using that $\langle\psi|\,\hat{\Delta A}^\dagger\hat{\Delta A}\,|\psi\rangle = \langle\psi|\,\hat{\Delta A}^2\,|\psi\rangle = \langle\hat{\Delta A}^2\rangle_\psi$, according to the definition (6.28).

It will be important to note the conditions under which (B.46) holds with equality. For this, both inequalities employed in the proof must hold with equality. The triangle inequality becomes an equality if $z^* = -z$, *i.e.* if $z \in \mathrm{i}\mathbb{R}$, that is if

$$\langle\psi|\, \Delta\hat{B}^\dagger \Delta\hat{A}\, |\psi\rangle \in \mathrm{i}\mathbb{R}. \tag{B.48}$$

The Cauchy-Schwarz inequality holds with equality if $|\phi\rangle$ and $|\phi'\rangle$ are collinear, that is if there exist $\alpha, \beta \in \mathbb{C}$ not both zero, such that

$$\alpha\Delta\hat{A}\,|\psi\rangle = \beta\Delta\hat{B}\,|\psi\rangle\,. \tag{B.49}$$

With this second condition, the first condition (B.48) simplifies to

$$\alpha\beta^* \in \mathrm{i}\mathbb{R}. \tag{B.50}$$

Squeezed Coherent States Uniquely Saturate Heisenberg Uertainty. Writing $|\psi\rangle = \int \mathrm{d}x\, \psi(x)\,|x\rangle$ in position space, the conditions (B.49) and (B.50) required for equality in the uncertainty relation (6.30) become

$$\alpha(x - x_0)\psi(x) = \beta(-\mathrm{i}\partial_x - p_0)\psi(x), \qquad \alpha\beta^* \in \mathrm{i}\mathbb{R}, \tag{B.51}$$

where x_0 and p_0 are the position and momentum expectation values of $|\psi\rangle$. Solving this differential equation gives

$$\beta \ln \psi(x) = \mathrm{i}\alpha\, \frac{(x - x_0)^2}{2} + \beta \mathrm{i} p_0 x + C, \tag{B.52}$$

where $C \in \mathbb{C}$ is an integration constant.

We will for now assume that α and β are *both* non-zero, but come back to the cases $\alpha = 0$ and $\beta = 0$ later. This allows us without loss of generality to take $\beta = 1$ and $\alpha = \mathrm{i}/\omega^2$, with $\omega^2 \in \mathbb{R}$ (the reason for the square will become clear shortly). This yields

$$\psi(x) = \mathrm{e}^C \exp\left(-\frac{(x - x_0)^2}{2\omega^2} + \mathrm{i}p_0 x\right). \tag{B.53}$$

Note that in order for ψ to remain finite at infinity requires $\omega^2 > 0$, so we will assume this from now on (this also explains the square in our notation). We then choose $\exp(C) > 0$ such that the state is normalized, *i.e.* $\langle\psi|\psi\rangle = \int \mathrm{d}x\, |\psi(x)|^2 = 1$, and obtain the family of normalized x-p-squeezed coherent wave-functions:

$$\psi^\omega_{x_0,p_0}(x) := \frac{1}{\sqrt{\omega\sqrt{\pi}}} \exp\left(-\frac{(x - x_0)^2}{2\omega^2} + \mathrm{i}p_0 x\right). \tag{B.54}$$

This is (6.24) in the main text and shows that squeezed coherent states are the only proper states which saturate (6.30).

Let us now recover some of the cases which we excluded earlier. Firstly, it makes little sense to include states with $\omega < 0$, since they exponentially diverge towards infinity, and we would expect $\psi(x)$ (and $\psi(p)$) to at least remain finite for $|x| \to \infty$ ($|p| \to \infty$ respectively). Secondly, the case $\alpha = 0$ can be reached by the limit $\omega \to \infty$, most easily computed in momentum space to yield a momentum eigenstate centred on p_0:

$$\lim_{\omega\to\infty} \psi^\omega_{x_0,p_0}(p) \propto \delta(p_0). \tag{B.55}$$

Thirdly, the case $\beta = 0$ can be reached by the limit $\omega \to 0$; we see that this results in a position eigenstate centred on x_0:

$$\lim_{\omega\to 0} \psi^\omega_{x_0,p_0}(x) \propto \delta(x_0). \tag{B.56}$$

These are (6.32) and (6.31) in the main text. Finally, we may always add a phase to any of the wave functions without changing the fact that their states are squeezed coherent; this completely restores the freedom in choosing C.

List of Figures

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